STEP Project

A Solution Booklet to STEP questions

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The PDF file can be found here, with all the generated files here.
The PDF files split up by year, by paper and by question are available as well.
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Notice that

$$I_n = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(x^2+2ax+b)^n} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{((x+a)^2+(b-a^2))^n}.$$

1. Let $x + a = \sqrt{b-a^2}\tan u$. When $x \to -\infty$, $u \to -\frac{\pi}{2}$, and when $x \to +\infty$, $u \to \frac{\pi}{2}$. We have also

$$\begin{aligned}\mathrm{d}x = \mathrm{d}(x+a) &= \mathrm{d}\sqrt{b-a^2}\tan u \\ &= \sqrt{b-a^2}\,\mathrm{d}\tan u \\ &= \sqrt{b-a^2}\sec^2 u\,\mathrm{d}u.\end{aligned}$$

   Therefore, we have

$$\begin{aligned}I_1 &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(x+a)^2+(b-a^2)} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b-a^2}\sec^2 u\,\mathrm{d}u}{\left(\sqrt{b-a^2}\tan u\right)^2+(b-a^2)} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b-a^2}\sec^2 u\,\mathrm{d}u}{(b-a^2)(\tan^2 u+1)} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sec^2 u\,\mathrm{d}u}{\sqrt{b-a^2}\sec^2 u} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sqrt{b-a^2}} \\ &= \frac{\pi}{\sqrt{b-a^2}},\end{aligned}$$

   as desired.

2. Using the same substitution, we have

$$\begin{aligned}I_n &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{[(x+a)^2+(b-a^2)]^n} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b-a^2}\sec^2 u\,\mathrm{d}u}{[(b-a^2)\sec^2 u]^n} \\ &= \frac{1}{\sqrt{b-a^2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{[(b-a^2)\sec^2 u]^{n-1}}.\end{aligned}$$

   Therefore,

$$2n(b-a^2)I_{n+1} = (2n-1)I_n,$$

   is equivalent to

$$2n\sqrt{b-a^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{[(b-a^2)\sec^2 u]^n} = (2n-1)\frac{1}{\sqrt{b-a^2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{[(b-a^2)\sec^2 u]^{n-1}}$$

   is equivalent to

$$2n(b-a^2)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{[(b-a^2)\sec^2 u]^n} = (2n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{[(b-a^2)\sec^2 u]^{n-1}}$$

   is equivalent to

$$2n\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n} u} = (2n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n-2} u}.$$

Notice that

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n-2} u} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sec^2 u \,\mathrm{d}u}{\sec^{2n} u} \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}\tan u}{\sec^{2n} u} \\
&= \lim_{\substack{a\to\frac{\pi}{2}\\ b\to-\frac{\pi}{2}}} \left[\frac{\tan u}{\sec^{2n} u}\right]_b^a - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan u \,\mathrm{d}\sec^{-2n} u \\
&= \lim_{\substack{a\to\frac{\pi}{2}\\ b\to-\frac{\pi}{2}}} \left[\sin u \cos^{2n-1} u\right]_b^a - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -2n \sec u \tan u \sec^{-2n-1} u \tan u \,\mathrm{d}u \\
&= 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\tan^2 u \,\mathrm{d}u}{\sec^{2n} u} \\
&= 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\sec^2 u - 1)\,\mathrm{d}u}{\sec^{2n} u} \\
&= 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n-2} u} - 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n} u}.
\end{aligned}$$

This means

$$(2n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n-2} u} = 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n} u},$$

which is exactly what was desired.

3. Proof by induction:

- **Base Case.** When $n = 1$,

$$\text{LHS} = I_1 = \frac{\pi}{\sqrt{b-a^2}},$$

$$\text{RHS} = \frac{\pi}{2^{2\cdot 1-2}(b-a^2)^{1-\frac{1}{2}}} \binom{2\cdot 1 - 2}{1-1} = \frac{\pi}{\sqrt{b-a^2}} \binom{0}{0} = \frac{\pi}{\sqrt{b-a^2}}.$$

- **Induction Hypothesis.** Assume for some $n = k \in \mathbb{N}$, we have

$$I_n = \frac{\pi}{2^{2n-2}(b-a^2)^{n-\frac{1}{2}}} \binom{2n-2}{n-1}.$$

- **Induction Step.** When $n = k + 1$,

$$\begin{aligned}
I_n &= I_{k+1} \\
&= \frac{2k+1}{2(k+1)(b-a^2)} I_k \\
&= \frac{2k+1}{2(k+1)(b-a^2)} \cdot \frac{\pi}{2^{2k-2}(b-a^2)^{k-\frac{1}{2}}} \binom{2k-2}{k-1} \\
&= \frac{\pi}{2^{2k}(b-a^2)^{k+\frac{1}{2}}} \frac{(2k-2)!}{(k-1)!(k-1)!} \frac{(2k+1)(2k+2)}{(k+1)^2} \\
&= \frac{\pi}{2^{2k}(b-a^2)^{k+\frac{1}{2}}} \frac{2k!}{k!k!} \\
&= \frac{\pi}{2^{2k}(b-a^2)^{k+\frac{1}{2}}} \binom{2k}{k} \\
&= \frac{\pi}{2^{2n-2}(b-a^2)^{n-\frac{1}{2}}} \binom{2n-2}{n-1}.
\end{aligned}$$

Therefore, by the principle of mathematical induction, for $n \in \mathbb{N}$,

$$I_n = \frac{\pi}{2^{2n-2}(b-a^2)^{n-\frac{1}{2}}} \binom{2n-2}{n-1},$$

as desired.

2016.3 Question 2

1. For $y^2 = 4ax$, we have $x = \frac{y^2}{4a}$, and therefore
$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{2y}{4a}.$$
Therefore, the normal through Q, l_Q satisfies that
$$l_Q : x - aq^2 = -\frac{4a}{2 \cdot 2aq} \cdot (y - 2aq),$$
i.e.
$$l_Q : q(x - aq^2) = -(y - 2aq).$$
Since $P \in l_Q$, we must have
$$\begin{aligned} q(ap^2 - aq^2) &= -(2ap - 2aq) \\ aq(p+q)(p-q) &= -2a(p-q) \\ pq + q^2 &= -2 \\ q^2 + pq + 2 &= 0 \end{aligned}$$
as desired.

2. We also have
$$r^2 + pr + 2 = 0.$$
Since $q \neq r$, q, r are the solutions to the equation
$$x^2 + px + 2 = 0,$$
and therefore $q + r = -p, qr = 2$.
Note that the equation for QR satisfies that
$$m_{QR} = \frac{2ar - 2aq}{ar^2 - aq^2} = \frac{2}{r+q}.$$
Therefore, l_{QR} satisfies that
$$\begin{aligned} l_{QR} : y - 2aq &= \frac{2}{r+q}(x - aq^2) \\ y &= \frac{2}{r+q}\left(x - aq^2 + \frac{r+q}{2} \cdot 2aq\right) \\ y &= \frac{2}{r+q}\left(x - aq^2 + aq^2 + aqr\right) \\ y &= \frac{2}{r+q}\left(x + aqr\right) \\ y &= -\frac{2}{p}(x + 2a). \end{aligned}$$
This passes through a fixed point $(-2a, 0)$.

3. OP has equation $y = \frac{2ap}{ap^2}x$, which is $y = \frac{2x}{p}$. Therefore, since $T = OP \cap QR$, x_T must satisfy that
$$\begin{aligned} -\frac{2}{p}(x + 2a) &= \frac{2x}{p}, \\ -(x + 2a) &= x \\ x &= -a. \end{aligned}$$
Therefore, $y_T = -\frac{2a}{p}$, $T\left(-a, -\frac{2a}{p}\right)$ lies on the line $x = -a$ which is independent of p.

The distance from the x-axis to T is $\left|\frac{2a}{p}\right| = \frac{2a}{|p|}$.

Notice that since $qr = 2$, q and r must take the same parity, and therefore $|p| = |q| + |r|$. By the AM-GM inequality, we have

$$|q| + |r| \geq 2\sqrt{|q| \cdot |r|} = 2\sqrt{2},$$

with the equal sign holding if and only if $|q| = |r|$, $q = r$, which is impossible.

Therefore, $|p| > 2\sqrt{2}$ and therefore $\frac{2a}{|p|} < \sqrt{2}$ as desired.

2016.3 Question 3

1. We have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\frac{e^x P(x)}{Q(x)} &= \frac{Q(x)\left[e^x P'(x) + e^x P(x)\right] - Q'(x)e^x P(x)}{Q(x)^2} \\ &= e^x \frac{\left[Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)\right]}{Q(x)^2} \\ &= e^x \frac{x^3-2}{(x+1)^2}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{\left[Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)\right]}{Q(x)^2} &= \frac{x^3-2}{(x+1)^2} \\ (x+1)^2\left[Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)\right] &= Q(x)^2\left(x^3-2\right).\end{aligned}$$

If we plug in $x = -1$ on both sides, we have $\text{LHS} = 0$ and $\text{RHS} = Q(-1)^2 \cdot (-3)$.
Therefore, $Q(-1)^2 = 0$, $Q(-1) = 0$.
Since $Q(x) \in \mathbb{P}[x]$, we must have

$$(x+1) \mid Q(x)$$

as desired.
Therefore, $\deg Q \geq 1$, $\deg \text{RHS} = 3 + 2\deg Q$.
If $\deg P = -\infty$, $P(x) = 0$,$\text{LHS} = 0$ which is impossible.
If $\deg P = 0$, $P(x) = C \in \mathbb{R}\setminus\{0\}$, $\text{LHS} = C(x+1)^2 Q(x)$, $\deg \text{LHS} = \deg q + 2$, which is impossible.
Therefore, we have $\deg P' = \deg P - 1$. Hence,

$$\deg Q(x)P'(x) = \deg P'(x)Q(x) = \deg P + \deg Q - 1,$$

and

$$\deg Q(x)P(x) = \deg P + \deg Q.$$

Therefore,

$$\deg \text{LHS} = 2 + \deg P + \deg Q = \deg \text{RHS},$$

which gives

$$\deg P = \deg Q + 1,$$

as desired.
When $Q(x) = x+1$, let $P(x) = ax^2 + bx + c$ where $a \neq 0$. We have $P'(x) = 2ax + b$. Therefore,

$$\begin{aligned}(x+1)^2\left[Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)\right] &= Q(x)^2\left(x^3-2\right) \\ Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x) &= x^3 - 2 \\ (x+1)(2ax+b) + (x+1)(ax^2+bx+c) - (ax^2+bx+c) &= x^3 - 2 \\ (x+1)(2ax+b) + x(ax^2+bx+c) &= x^3 - 2 \\ ax^3 + (2a+b)x^2 + (2a+b+c)x + b &= x^3 - 2.\end{aligned}$$

This solves to $(a, b, c) = (1, -2, 0)$. Therefore, $P(x) = x^2 - 2x$.

2. In this case, we must have that

$$(x+1)\left[Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)\right] = Q(x)^2.$$

Therefore, $Q(x) = (x+1)R(x)$ for some $R(x) \in \mathbb{P}[x]$. We may assume $P(-1) \neq 0$.
Hence, $Q'(x) = (x+1)R'(x) + R(x)$

Plugging this in gives us

$$(x+1)R(x)P'(x) + (x+1)R(x)P(x) - \left[(x+1)R'(x) + R(x)\right]P(x) = (x+1)R(x)^2,$$

which simplifies to

$$(x+1)\left[R(x)P'(x) + R(x)P(x) - R'(x)P(x)\right] - R(x)P(x) = (x+1)R(x)^2.$$

Let $x = -1$, and we can see $x+1$ divides $R(x)$, since $x+1$ can't divide $P(x)$.

Therefore, let $R(x) = (x+1)S(x)$, therefore $R'(x) = S(x) + (x+1)S'(x)$.

This gives

$$(x+1)S(x)\left[P'(x) + P(x)\right] - \left[S(x) + (x+1)S'(x)\right]P(x) - S(x)P(x) = (x+1)^2 S(x)^2,$$

which simplifies to

$$(x+1)\left[S(x)P'(x) + S(x)P(x) - S'(x)P(x)\right] - 2S(x)P(x) = (x+1)^2 S(x)^2.$$

Therefore, we can see that $x+1$ divides $S(x)$ by similar reasons.

Repeating this, we can conclude that there are arbitrarily many factors of $x+1$ in $Q(x)$ (proof by infinite descent), which is impossible.

Formally speaking, let $Q(x) = (x+1)^n T(x)$ where $T(-1) \neq 0$, $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} Q'(x) &= n(x+1)^{n-1}T(x) + (x+1)^n T'(x) \\ &= (x+1)^{n-1}\left[nT(x) + (x+1)T'(x)\right]. \end{aligned}$$

Therefore,

$$(x+1)\left[Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)\right] = Q(x)^2$$

simplifies to

$$(x+1)^{n+1}T(x)\left[P'(x) + P(x)\right] - (x+1)^n\left[nT(x) + (x+1)T'(x)\right]P(x) = (x+1)^{2n}T(x)^2,$$

which further simplifies to

$$(x+1)\left[T(x)P'(x) + T(x)P(x) - T'(x)P(x)\right] - nT(x)P(x) = (x+1)^n T(x)^2.$$

Now, let $x = -1$, we have that $nT(-1)P(-1) = 0$. But $n \neq 0$, $T(-1) \neq 0$, $P(-1) \neq 0$, which gives a contradiction.

Therefore, such P and Q do not exist.

2016.3 Question 4

1. Notice that
$$\frac{1}{1+x^r}-\frac{1}{1+x^{r+1}}=\frac{x^{r+1}-x^r}{(1+x^r)(1+x^{r+1})}=\frac{x^r(x-1)}{(1+x^r)(1+x^{r+1})}.$$

   Therefore, we have
$$\begin{aligned}\sum_{r=1}^{N}\frac{x^r}{(1+x^r)(1+x^{r+1})}&=\sum_{r=1}^{N}\frac{1}{x-1}\left[\frac{1}{1+x^r}-\frac{1}{1+x^{r+1}}\right]\\&=\frac{1}{x-1}\sum_{r=1}^{N}\left[\frac{1}{1+x^r}-\frac{1}{1+x^{r+1}}\right]\\&=\frac{1}{x-1}\left[\frac{1}{1+x}-\frac{1}{1+x^{n+1}}\right].\end{aligned}$$

   For $|x|<1$, as $n\to\infty$, $x^{n+1}\to 0$. Therefore,
$$\begin{aligned}\sum_{r=1}^{\infty}\frac{x^r}{(1+x^r)(1+x^{r+1})}&=\frac{1}{x-1}\left[\frac{1}{1+x}-1\right]\\&=\frac{1}{x-1}\cdot\frac{-x}{1+x}\\&=\frac{x}{1-x^2}\end{aligned}$$

   as desired.

2. Notice that
$$\begin{aligned}\operatorname{sech}(ry)\operatorname{sech}((r+1)y)&=\frac{2}{e^{ry}+e^{-ry}}\cdot\frac{2}{e^{(r+1)y}+e^{-(r+1)y}}\\&=\frac{4e^{-ry-(r+1)y}}{(1+e^{-2ry})\left(1+e^{-2(r+1)y}\right)}\\&=4e^{-y}\frac{e^{-2ry}}{(1+e^{-2ry})\left(1+e^{-2(r+1)y}\right)}.\end{aligned}$$

   Let $x=e^{-2y}$. We have
$$\operatorname{sech}(ry)\operatorname{sech}((r+1)y)=4e^{-y}\frac{x^r}{(1+x^r)(1+x^{r+1})}.$$

   When $y>0$, $x=e^{-2y}\in(0,1)$. Therefore,
$$\begin{aligned}\sum_{r=1}^{\infty}\operatorname{sech}(ry)\operatorname{sech}((r+1)y)&=4e^{-y}\frac{e^{-2y}}{1-e^{-4y}}\\&=2e^{-y}\frac{2}{e^{2y}-e^{-2y}}\\&=2e^{-y}\operatorname{cosech}(2y)\end{aligned}$$

   as desired.

   Notice that for all $x\in\mathbb{R}$, $\cosh x=\cosh(-x)$, therefore $\operatorname{sech} x=\operatorname{sech}(-x)$.

Therefore,

$$\begin{aligned}
&\sum_{r=-\infty}^{\infty} \operatorname{sech}(ry) \operatorname{sech}((r+1)y) \\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry) \operatorname{sech}((r+1)y) + \sum_{r=-\infty}^{0} \operatorname{sech}(ry) \operatorname{sech}((r+1)y) \\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry) \operatorname{sech}((r+1)y) + \sum_{r=0}^{+\infty} \operatorname{sech}(-ry) \operatorname{sech}((-r+1)y) \\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry) \operatorname{sech}((r+1)y) + \sum_{r=0}^{+\infty} \operatorname{sech}(ry) \operatorname{sech}((r-1)y) \\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry) \operatorname{sech}((r+1)y) + \sum_{r=2}^{+\infty} \operatorname{sech}(ry) \operatorname{sech}((r-1)y) + \operatorname{sech}(y) \operatorname{sech}(0) + \operatorname{sech}(0) \operatorname{sech}(-y) \\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry) \operatorname{sech}((r+1)y) + \sum_{r=1}^{+\infty} \operatorname{sech}((r+1)y) \operatorname{sech}(ry) + 2 \operatorname{sech} y \\
&= 4e^{-y} \operatorname{cosech}(2y) + 2 \operatorname{sech} y \\
&= \frac{4e^{-y}}{\sinh 2y} + \frac{2}{\cosh y} \\
&= \frac{2e^{-y}}{\sinh y \cosh y} + \frac{2}{\cosh y} \\
&= \frac{2e^{-y} + 2 \sinh y}{\sinh y \cosh y} \\
&= \frac{2e^{-y} + e^{y} - e^{-y}}{\sinh y \cosh y} \\
&= \frac{e^{y} - e^{-y}}{\sinh y \cosh y} \\
&= \frac{2 \cosh y}{\sinh y \cosh y} \\
&= 2 \operatorname{cosech} y.
\end{aligned}$$

2016.3 Question 5

1. By the binomial theorem, we have

$$(1+x)^{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} x^k.$$

If we let $x = 1$, we have

$$2^{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k}.$$

Since $\binom{2m+1}{m}$ is a part of the sum, and all the other terms are positive, and there are other terms which are not $\binom{2m+1}{m}$ (e.g. $\binom{2m+1}{0} = 1$), we therefore must have

$$\binom{2m+1}{m} < 2^{2m+1}.$$

2. Notice that

$$\begin{aligned}\binom{2m+1}{m} &= \frac{(2m+1)!}{m!(m+1)!}\\ &= \frac{(2m+1)(2m)(2m-1)\cdots(m+2)}{m!}\end{aligned}$$

A number theory argument follows. First, notice that all terms in the product $P_{m+1,2m+1}$ are within the numerator. Therefore, we must have

$$P_{m+1,2m+1} \mid (2m+1)(2m)(2m-1)\cdots(m+2).$$

Next, since all the terms in the product are primes, none of the terms will therefore have factors between 1 and m. This means that

$$\gcd\left(P_{m+1,2m+1}, m!\right) = 1,$$

i.e. $P_{m+1,2m+1}$ are coprime.

Therefore, given that $\binom{2m+1}{m} = \frac{(2m+1)(2m)(2m-1)\cdots(m+2)}{m!}$ is an integer, we must therefore have

$$P_{m+1,2m+1} \mid \binom{2m+1}{m},$$

and hence

$$P_{m+1,2m+1} \leq \binom{2m+1}{m} < 2^{2m},$$

as desired.

3. Notice that

$$\begin{aligned}P_{1,2m+1} &= P_{1,m+1} \cdot P_{m+1,2m+1}\\ &< 4^{m+1} \cdot 2^{2m}\\ &= 4^{m+1} \cdot 4^m\\ &= 4^{2m+1},\end{aligned}$$

as desired.

4. First we look at the base case when $n = 2$.

   $P_{1,2} = 2$, $4^2 = 16$, the original statement holds when $n = 2$.

   Now, we use strong induction. Suppose the statement holds up to some $n = k \geq 2$.

   If $k = 2m$ is even, the induction step for $2m \to 2m + 1$ is already shown in the previous part.

   If $k = 2m + 1$ is odd, we must have that $k + 1$ is even. The only even prime is 2, but since $k \geq 2$, $k + 1 \neq 2$, and $k + 1$ must be composite.

   Therefore, $P_{1,k+1} = P_{1,k} < 4^k < 4^{k+1}$. This completes the induction step.

   Therefore, by strong induction, the statement $P_{1,n} < 4^n$ holds for all $n \geq 2$.

2016.3 Question 6

- In the case where $B > A > 0$ or $-B < -A < 0$, notice that

$$R\cosh(x+\gamma) = R\cosh x\cosh\gamma + R\sinh x\sinh\gamma.$$

  Therefore, we would like $R\sinh\gamma = A$ and $R\cosh\gamma = B$.
  Since $\cosh\gamma^2 - \sinh\gamma^2 = 1$, we have $R^2 = B^2 - A^2$.
  We also have $\tanh\gamma = \frac{A}{B}$, and therefore $\gamma = \operatorname{artanh}\frac{A}{B}$.
  Notice that $\cosh\gamma > 0$, so R must have the same sign as B.

  - If $B > A > 0$, $R = \sqrt{B^2 - A^2}$.
  - If $B < -A < 0$, $R = -\sqrt{B^2 - A^2}$.

- In the case where $-A < B < A$, notice that

$$R\sinh(x+\gamma) = R\sinh\gamma\cosh x + R\cosh\gamma\sinh x.$$

  Therefore, we would like $R\cosh\gamma = A$ and $R\sinh\gamma = B$.
  Since $\cosh\gamma^2 - \sinh\gamma^2 = 1$, we have $R^2 = B^2 - A^2$.
  We also have $\tanh\gamma = \frac{B}{A}$, and therefore $\gamma = \operatorname{artanh}\frac{B}{A}$.
  Notice that $\cosh\gamma > 0$, so R will have the same sign as A, and hence $R = \sqrt{A^2 - B^2}$.

- When $B = A$, we have

$$\begin{aligned} A\sinh x + B\cosh x &= A\frac{e^x - e^{-x}}{2} + A\frac{e^x + e^{-x}}{2} \\ &= Ae^x. \end{aligned}$$

- When $B = -A$, we have

$$\begin{aligned} A\sinh x + B\cosh x &= A\frac{e^x - e^{-x}}{2} - A\frac{e^x + e^{-x}}{2} \\ &= Ae^{-x}. \end{aligned}$$

Therefore, in conclusion,

$$A\sinh x + B\cosh x = \begin{cases} \sqrt{B^2 - A^2}\cosh\left(x + \operatorname{artanh}\frac{A}{B}\right), & 0 < A < B, \\ Ae^x, & 0 < B = A, \\ \sqrt{A^2 - B^2}\sinh\left(x + \operatorname{artanh}\frac{B}{A}\right), & -A < B < A, \\ -Ae^{-x}, & B = -A < 0, \\ -\sqrt{B^2 - A^2}\cosh\left(x + \operatorname{artanh}\frac{A}{B}\right), & -B < -A < 0. \end{cases}$$

1. We have $\operatorname{sech} x = a\tanh x + b$, and hence $1 = a\sinh x + b\cosh x$. If $b > a > 0$, we have

$$\sqrt{b^2 - a^2}\cosh\left(x + \operatorname{artanh}\frac{a}{b}\right) = 1.$$

   Therefore,

$$\begin{aligned} \cosh\left(x + \operatorname{artanh}\frac{a}{b}\right) &= \frac{1}{\sqrt{b^2 - a^2}} \\ x + \operatorname{artanh}\frac{a}{b} &= \pm\operatorname{arcosh}\frac{1}{\sqrt{b^2 - a^2}} \\ x &= \pm\operatorname{arcosh}\frac{1}{\sqrt{b^2 - a^2}} - \operatorname{artanh}\frac{a}{b}, \end{aligned}$$

   as desired.

2. When $a > b > 0$,
$$\sqrt{a^2 - b^2} \sinh\left(x + \operatorname{artanh} \frac{b}{a}\right) = 1.$$
Therefore,
$$\begin{aligned}\sinh\left(x + \operatorname{artanh} \frac{b}{a}\right) &= \frac{1}{\sqrt{a^2 - b^2}} \\ x + \operatorname{artanh} \frac{b}{a} &= \operatorname{arsinh} \frac{1}{\sqrt{a^2 - b^2}} \\ x &= \operatorname{arsinh} \frac{1}{\sqrt{a^2 - b^2}} - \operatorname{artanh} \frac{b}{a}.\end{aligned}$$

3. We would like to have two solutions to the equation $1 = a \sinh x + b \cosh x$.
   - $0 < a < b$, this gives
   $$x = \pm \operatorname{arcosh} \frac{1}{\sqrt{b^2 - a^2}} - \operatorname{artanh} \frac{a}{b},$$
   For this to make sense, we must have $\frac{1}{\sqrt{b^2 - a^2}} \geq 1$, and therefore $0 < \sqrt{b^2 - a^2} \leq 1$, which is $0 < b^2 - a^2 \leq 1$.
   For this to have two distinct points, we would like to have $\operatorname{arcosh} \frac{1}{\sqrt{b^2 - a^2}} \neq 0$ as well. This means $b^2 - a^2 \neq 1$.
   Therefore, in this case, this means that $a < b < \sqrt{a^2 + 1}$.
   - $b = a$, this gives $ae^x = 1$, which gives a unique solution $x = -\ln a$.
   - $-a < b < a$, this gives
   $$\sqrt{A^2 - B^2} \sinh\left(x + \operatorname{artanh} \frac{B}{A}\right) = 1,$$
   which can only give the solution $x = \operatorname{arsinh} \frac{1}{\sqrt{A^2 - B^2}} - \operatorname{artanh} \frac{B}{A}$.
   - $b = -a$, this gives $-ae^{-x} = 1$, which does not have a solution.
   - $-b < -a < 0$, this gives
   $$-\sqrt{b^2 - a^2} \cosh\left(x + \operatorname{artanh} \frac{a}{b}\right) = 1,$$
   but this is impossible, since both square root and cosh are always positive.

   Therefore, the only possibility is when $a < b < \sqrt{a^2 + 1}$.

4. When they touch at a point, this will mean at this value, the number of solutions will change on both sides. This is only possible when $b = \sqrt{a^2 + 1}$.
   Therefore,
   $$x = -\operatorname{artanh} \frac{a}{\sqrt{a^2 + 1}}.$$
   Hence,
   $$\begin{aligned}y &= a \tanh x + b \\ &= -a \cdot \frac{a}{\sqrt{a^2 + 1}} + \sqrt{a^2 + 1} \\ &= \frac{-a^2 + a^2 + 1}{\sqrt{a^2 + 1}} \\ &= \frac{1}{\sqrt{a^2 + 1}}.\end{aligned}$$

2016.3 Question 7

For $\omega = \exp \frac{2\pi i}{n}$, we have for $k = 0, 1, 2, \ldots, n-1$, that $\omega^k = \exp \frac{2\pi i k}{n}$. Therefore,

$$(\omega^k)^n = \exp \frac{2\pi i k n}{n} = \exp(2\pi i k) = 1.$$

Also, notice that $\arg \omega^k = \frac{2k\pi}{n}$, which means that all ω^ks are different.

This means that $\omega^0 = 1, \omega^1 = 1, \omega^2, \ldots, \omega^{n-1}$ are exactly the n roots to the polynomial $z^n - 1$, which has leading coefficient 1.

Therefore, we must have

$$(z-1)(z-\omega)\cdots(z-\omega^{n-1}) = z^n - 1,$$

as desired.

For the following parts, W.L.O.G. let the orientation of the polygon be such that $X_k = \omega^k$.

1. Let z represent the complex number for P, we have

$$\begin{aligned}\prod_{k=0}^{n-1} |PX_k| &= \prod_{k=0}^{n-1} \left|z - \omega^k\right| \\ &= \left|\prod_{k=0}^{n-1} (z - \omega^k)\right| \\ &= \left|z^n - 1\right|.\end{aligned}$$

Since P is equidistant from X_0 and X_1, we must have that $P = r \exp\left(\frac{\pi i}{n}\right)$ for some $r \in \mathbb{R}$, where $|r| = |OP|$. Therefore, we have

$$\begin{aligned}\prod_{k=0}^{n-1} |PX_k| &= |z^n - 1| \\ &= \left|r^n \exp\left(\frac{\pi i}{2}\right) - 1\right| \\ &= |-r^n - 1| \\ &= \left|r^n + 1\right|.\end{aligned}$$

If n is even, then $r^n = |r|^n > 0$, and therefore $|r^n + 1| = r^n + 1 = |r|^n + 1 = |OP|^n + 1$ as desired.

If n is odd, and $r > 0$, then $r^n = |r|^n > 0$, and

$$\begin{aligned}\text{LHS} &= |r^n + 1| \\ &= r^n + 1 \\ &= |r|^n + 1 \\ &= |OP|^n + 1.\end{aligned}$$

When $-1 \leq r < 0$, we have $-1 \leq r^n = -|r|^n < 0$, and

$$\begin{aligned}\text{LHS} &= |r^n + 1| \\ &= r^n + 1 \\ &= -|r|^n + 1 \\ &= -|OP|^n + 1.\end{aligned}$$

When $r < -1$, we have $r^n = -|r|^n < -1$, and

$$\begin{aligned}\text{LHS} &= |r^n + 1| \\ &= -r^n - 1 \\ &= |r|^n - 1 \\ &= |OP|^n - 1.\end{aligned}$$

In summary, when n is odd, we have

$$\prod_{k=0}^{n-1} |PX_k| = \begin{cases} |OP|^n + 1, & P \text{ is in the first quadrant,} \\ -|OP|^n + 1, & P \text{ is in the third quadrant and } |OP| \leq 1, \\ |OP|^n - 1, & P \text{ is in the third quadrant and } |OP| > 1. \end{cases}$$

2. Notice that for a general point P whose complex number is z, we have

$$\begin{aligned} \prod_{k=1}^{n-1} |PX_k| &= (z-\omega)(z-\omega^2)\cdots\left(z-\omega^{n-1}\right) \\ &= \frac{z^n - 1}{z-1} \\ &= 1 + z + z^2 + \cdots + z^{n-1}. \end{aligned}$$

If we let $P = X_0$, $z = 1$, and RHS $= n$, just as we desired.

2016.3 Question 8

1. If we replace x with $-x$ in the original equation, we get
$$f(-x) + (1 - (-x))f(-(-x)) = (-x)^2,$$
which simplifies to
$$f(-x) + (1 + x)f(x) = x^2$$
as desired.
Therefore, we have a pair of equations in terms of $f(x)$ and $f(-x)$:
$$\begin{cases} f(x) + (1 - x)f(-x) & = x^2 \\ (1 + x)f(x) + f(-x) & = x^2. \end{cases}$$
Multiplying the second equation by $(1 - x)$ gives us
$$(1 - x^2)f(x) + (1 - x)f(-x) = x^2(1 - x),$$
and subtracting the first equation from this
$$-x^2 f(x) = -x^3,$$
which gives $f(x) = x$.
Plugging this back, we have
$$\begin{aligned} \text{LHS} &= f(x) + (1 - x)f(-x) \\ &= x + (1 - x)(-x) \\ &= x - x + x^2 \\ &= x^2 \\ &= \text{RHS} \end{aligned}$$
which holds. Therefore, $f(x) = x$ is the solution to the functional equation.

2. For $x \neq 1$, we have
$$\begin{aligned} K(K(x)) &= \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} \\ &= \frac{(x + 1) + (x - 1)}{(x + 1) - (x - 1)} \\ &= \frac{2x}{2} \\ &= x, \end{aligned}$$
for $x \neq 1$, as desired.
The equation on g is
$$g(x) + xg(K(x)) = x,$$
and if we substitute x as $K(x)$, we have
$$g(K(x)) + K(x)g(K(K(x))) = K(x),$$
which simplifies to
$$g(K(x)) + K(x)g(x) = K(x).$$
Multiplying the second equation by x, we have
$$xK(x)g(X) + xg(K(x)) = xK(x),$$

and subtracting the first equation from this gives

$$(xK(x) - 1)g(x) = x(K(x) - 1),$$

which gives

$$\begin{aligned} g(x) &= \frac{x\left(K(x) - 1\right)}{xK(x) - 1} \\ &= \frac{x\left(\frac{x+1}{x-1} - 1\right)}{x \cdot \frac{x+1}{x-1} - 1} \\ &= \frac{x\left[(x+1) - (x-1)\right]}{x(x+1) - (x-1)} \\ &= \frac{2x}{x^2+1}, \end{aligned}$$

for $x \neq 1$.

If we plug this back to the original equation, we have

$$\begin{aligned} \text{LHS} &= \frac{2x}{x^2+1} + x\frac{2 \cdot \frac{x+1}{x-1}}{\left(\frac{x+1}{x-1}\right)^2 + 1} \\ &= \frac{2x}{x^2+1} + \frac{2x \cdot (x+1) \cdot (x-1)}{(x+1)^2 + (x-1)^2} \\ &= \frac{2x}{x^2+1} + \frac{2x(x^2-1)}{2x^2+2} \\ &= \frac{2x}{x^2+1} + \frac{x(x^2-1)}{x^2+1} \\ &= \frac{x^3 - x + 2x}{x^2+1} \\ &= \frac{x(x^2+1)}{x^2+1} \\ &= x \\ &= \text{RHS}, \end{aligned}$$

so

$$g(x) = \frac{2x}{x^2+1}$$

is the solution to the original functional equation.

3. Let $H(x) = \frac{1}{1-x}$. Notice that

$$\begin{aligned} H(H(x)) &= \frac{1}{1 - \frac{1}{1-x}} \\ &= \frac{1-x}{1-x-1} \\ &= \frac{x-1}{x} \\ &= 1 - \frac{1}{x} \end{aligned}$$

and

$$\begin{aligned} H(H(H(x))) &= \frac{1}{1 - \left(1 - \frac{1}{x}\right)} \\ &= \frac{x}{1} \\ &= x. \end{aligned}$$

Now, if we replace all the x with $\frac{1}{1-x}$, we will get

$$h\left(\frac{1}{1-x}\right)+h\left(1-\frac{1}{x}\right)=1-\frac{1}{1-x}-\left(1-\frac{1}{x}\right),$$

and doing the same replacement again gives us

$$h\left(1-\frac{1}{x}\right)+h(x)=1-\left(1-\frac{1}{x}\right)-x.$$

Summing these two equations, together with the original equation, gives us that

$$2\cdot\left[h\left(\frac{1}{1-x}\right)+h\left(1-\frac{1}{x}\right)+h(x)\right]=3-2\cdot\left[x+\frac{1}{1-x}+\left(1-\frac{1}{x}\right)\right],$$

and therefore

$$h\left(\frac{1}{1-x}\right)+h\left(1-\frac{1}{x}\right)+h(x)=\frac{3}{2}-\left[x+\frac{1}{1-x}+\left(1-\frac{1}{x}\right)\right].$$

Subtracting the second equation from this, gives that

$$\begin{aligned}h(x)&=\left(\frac{3}{2}-\left[x+\frac{1}{1-x}+\left(1-\frac{1}{x}\right)\right]\right)-\left[1-\frac{1}{1-x}-\left(1-\frac{1}{x}\right)\right]\\&=\frac{1}{2}-x.\end{aligned}$$

Plugging this back to the original equation, we have

$$\begin{aligned}\text{LHS}&=\frac{1}{2}-x+\frac{1}{2}-\frac{1}{1-x}\\&=1-x-\frac{1}{1-x}\\&=\text{RHS},\end{aligned}$$

which satisfies the original functional equation. Therefore, the original equation solves to

$$h(x)=\frac{1}{2}-x.$$

2016.3 Question 12

1. Let $X \sim \mathrm{B}(100n, 0.2)$. We have $\mu = 100n \cdot 0.2 = 20n$, and $\sigma^2 = 100n \cdot 0.2 \cdot 0.8 = 16n$.

   We have that

$$\begin{aligned}\alpha &= \mathrm{P}(16n \leq X \leq 24n) \\ &= \mathrm{P}(|(X - 20n)| \leq 4n) \\ &= \mathrm{P}(|(X - \mu)| \leq \sigma\sqrt{n}) \\ &= 1 - \mathrm{P}(|(X - \mu)| > \sigma\sqrt{n}) \\ &\geq 1 - \frac{1}{\sqrt{n}^2} \\ &= 1 - \frac{1}{n},\end{aligned}$$

   as desired, where we applied the Chebyshev Inequality for $k = \sqrt{n} > 0$.

2. Let $X \sim \mathrm{Po}(n)$. Therefore, $\mu = \mathrm{E}(X) = n$, $\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{n}$. To show the desired inequality is equivalent to showing that

$$\frac{1 + n + \frac{n^2}{2!} + \cdot + \frac{n^{2n}}{(2n!)}}{e^n} \geq 1 - \frac{1}{n}.$$

   Notice that the left-hand side is simply $\mathrm{P}(0 \leq X \leq 2n)$. By the Chebyshev Inequality, we have

$$\begin{aligned}\mathrm{LHS} &= \mathrm{P}(0 \leq X \leq 2n) \\ &= \mathrm{P}(|X - \mu| \leq n) \\ &= \mathrm{P}(|X - \mu| \leq \sqrt{n}\sigma) \\ &= 1 - \mathrm{P}(|X - \mu| > \sqrt{n}\sigma) \\ &\geq 1 - \frac{1}{n} \\ &= \mathrm{RHS},\end{aligned}$$

   as desired, where we applied the Chebyshev Inequality for $k = \sqrt{n} > 0$.

2016.3 Question 13

For a random variable X with $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$, we have

$$\kappa(X) = \frac{\mathrm{E}\left[(X-\mu)^4\right]}{\sigma^4} - 3$$

We have $Y = X - a$. Therefore, $\mathrm{E}(Y) = \mu - a$ and $\mathrm{Var}(Y) = \sigma^2$.

$$\begin{aligned}
\kappa(Y) &= \frac{\mathrm{E}\left[(Y-(\mu-a))^4\right]}{\sigma^4} - 3 \\
&= \frac{\mathrm{E}\left[((X-a)-(\mu-a))^4\right]}{\sigma^4} - 3 \\
&= \frac{\mathrm{E}\left[(X-\mu)^4\right]}{\sigma^4} - 3 \\
&= \kappa(X),
\end{aligned}$$

as desired.

1. Let $X \sim \mathrm{N}(0, \sigma^2)$, $\mu = 0$. Notice that

$$\kappa(X) = \frac{\mathrm{E}(X^4)}{\sigma^4} - 3.$$

X has p.d.f.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Therefore,

$$\mathrm{E}(X^4) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^4 \exp\left(-\frac{x^2}{2\sigma^2}\right) \mathrm{d}x.$$

Now, consider using integration by parts. Notice that

$$\mathrm{d}\exp\left(-\frac{x^2}{2\sigma^2}\right) = -\frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x,$$

and therefore, using integration by parts, we have

$$\begin{aligned}
&\int x^4 \exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x \\
&= -\sigma^2 \int x^3 \,\mathrm{d}\exp\left(-\frac{x^2}{2\sigma^2}\right) \\
&= -\sigma^2 \left[x^3 \exp\left(-\frac{x^2}{2\sigma^2}\right) - \int \exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}(x^3)\right] \\
&= 3\sigma^2 \int x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x - \sigma^2 x^3 \exp\left(-\frac{x^2}{2\sigma^2}\right).
\end{aligned}$$

Therefore, considering the definite integral, we have

$$\begin{aligned}
\mathrm{E}(X^4) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^4 \exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x \\
&= \frac{\sigma}{\sqrt{2\pi}}\left[3\int_{-\infty}^{+\infty} x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right)\mathrm{d}x - \left[x^3 \exp\left(-\frac{x^2}{2\sigma^2}\right)\right]_{-\infty}^{+\infty}\right] \\
&= \frac{\sigma}{\sqrt{2\pi}}\left[3 \cdot \sigma\sqrt{2\pi} \cdot \sigma^2 - 0\right] \\
&= 3\sigma^4.
\end{aligned}$$

Therefore,

$$\kappa(X) = \frac{\mathrm{E}(X^4)}{\sigma^4} - 3 = \frac{3\sigma^4}{\sigma^4} - 3 = 0,$$

as desired.

An alternative solution exists using generating functions.

Recall that a general normal distribution $\mathrm{N}(\mu, \sigma^2)$ has MGF

$$M(t) = \exp(\mu t + \frac{\sigma^2}{2}t^2),$$

and hence

$$\begin{aligned} M_X(t) &= \exp\left(\frac{\sigma^2}{2}t^2\right) \\ &= 1 + \left(\frac{\sigma^2}{2}t^2\right) + \frac{\left(\frac{\sigma^2}{2}t^2\right)}{2!} + \dots. \end{aligned}$$

Therefore,

$$\mathrm{E}(X^4) = M_X^{(4)}(0) = \left(\frac{\sigma^2}{2}\right)^4 \cdot 4! = 3\sigma^4,$$

and the result follows.

2. Notice that

$$\begin{aligned} &T^4 \\ &= \sum_a \binom{4}{4} Y_a^4 + \sum_{a<b} \left[\binom{4}{1,3} Y_a Y_b^3 + \binom{4}{2,2} Y_a^2 Y_b^2\right] \\ &\quad + \sum_{a<b<c} \binom{4}{1,1,2} Y_a Y_b Y_c^2 + \sum_{a<b<c<d} \binom{4}{1,1,1,1} Y_a Y_b Y_c Y_d \\ &= \sum_a Y_a^4 + \sum_{a<b} (4Y_a Y_b^3 + 6Y_a^2 Y_b^2) + \sum_{a<b<c} 12 Y_a Y_b Y_c^2 + \sum_{a<b<c<d} 24 Y_a Y_b Y_c Y_d, \end{aligned}$$

where

$$\binom{n}{a_1, a_2, \dots, a_k} = \frac{n!}{a_1! a_2! \dots a_k!}, \sum_{i=1}^k a_i = n$$

stands for the multinomial coefficient.

Note that $\mathrm{E}(Y_r) = 0$ for any $r = 1, 2, \dots, n$. Therefore,

$$\begin{aligned} \mathrm{E}(Y_a Y_b^3) &= \mathrm{E}(Y_a)\,\mathrm{E}(Y_b^3) = 0, \\ \mathrm{E}(Y_a Y_b Y_c^2) &= \mathrm{E}(Y_a)\,\mathrm{E}(Y_b)\,\mathrm{E}(Y_c^2) = 0, \\ \mathrm{E}(Y_a Y_b Y_c Y_d) &= \mathrm{E}(Y_a)\,\mathrm{E}(Y_b)\,\mathrm{E}(Y_c)\,\mathrm{E}(Y_d) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{E}(T^4) &= \sum_a \mathrm{E}(Y_a^4) + \sum_{a<b} 6\,\mathrm{E}(Y_a^2 Y_b^2) \\ &= \sum_{r=1}^n \mathrm{E}(Y_r^4) + 6 \sum_{r=1}^{n-1} \sum_{s=r+1}^n \mathrm{E}(Y_a^2)\,\mathrm{E}(Y_b^2), \end{aligned}$$

as desired.

3. Let $Y_i = X_i - \mu$ for $i = 1, 2, \dots, n$, and $\mu = \mathrm{E}(X), \sigma^2 = \mathrm{Var}(X) = \mathrm{Var}(Y)$ with $\mathrm{E}(Y) = 0$ Therefore, let $T = \sum_i^n Y_i = \sum_i^n X_i - n\mu$, we must have $\mathrm{E}(T) = 0$ and $\mathrm{Var}(T) = n\sigma^2$.

But since the kurtosis remains constant with shifts, we must have that $\kappa(Y_i) = \kappa$, and

$$\kappa(T) = \kappa\left[\sum_i^n X_i\right].$$

Hence, we have

$$\begin{aligned}
\kappa\left[\sum_i^n X_i\right] &= \kappa(T) \\
&= \frac{\mathrm{E}(T^4)}{(n\sigma^2)^2} - 3 \\
&= \frac{\sum_{r=1}^n \mathrm{E}(Y_r^4) + 6\sum_{r=1}^{n-1}\sum_{s=r+1}^n \mathrm{E}(Y_a^2)\,\mathrm{E}(Y_b^2)}{n^2\sigma^4} - 3 \\
&= \frac{1}{n^2}\sum_{r=1}^n \frac{\mathrm{E}(Y_r^4)}{\sigma^4} + \frac{6}{n^2}\sum_{r=1}^{n-1}\sum_{s=r+1}^n \frac{\sigma^4}{\sigma^4} - 3 \\
&= \frac{1}{n^2} n \cdot (\kappa + 3) + \frac{6}{n^2}\binom{n}{2} - 3 \\
&= \frac{\kappa}{n} + \frac{3n + 3n(n-1) - 3n^2}{n^2} \\
&= \frac{\kappa}{n} + 0 \\
&= \frac{\kappa}{n},
\end{aligned}$$

as desired.

Year 2017

2017 Paper 3

2017.3 Question 1

1. We have

$$\begin{aligned}\text{RHS} &= \frac{r+1}{r}\left(\frac{1}{\binom{n+r-1}{r}} - \frac{1}{\binom{n+r}{r}}\right)\\ &= \frac{r+1}{r}\left(\frac{r!(n-1)!}{(n+r-1)!} - \frac{r!n!}{(n+r)!}\right)\\ &= \frac{r+1}{r}\left(\frac{r!(n-1)!(n+r)}{(n+r)!} - \frac{r!(n-1)!n}{(n+r)!}\right)\\ &= \frac{r+1}{r}\cdot\frac{r!(n-1)!(n+r) - r!(n-1)!n}{(n+r)!}\\ &= \frac{r+1}{r}\cdot\frac{r!(n-1)!r}{(n+r)!}\\ &= \frac{(r+1)!(n-1)!}{(n+r)!}\\ &= \binom{n+r}{r+1}\\ &= \text{LHS}\end{aligned}$$

as desired.

Therefore,

$$\begin{aligned}\sum_{n=1}^{+\infty}\frac{1}{\binom{n+r}{r+1}} &= \sum_{n=1}^{+\infty}\frac{r+1}{r}\left(\frac{1}{\binom{n+r-1}{r}} - \frac{1}{\binom{n+r}{r}}\right)\\ &= \frac{r+1}{r}\sum_{n=1}^{+\infty}\left(\frac{1}{\binom{n+r-1}{r}} - \frac{1}{\binom{n+r}{r}}\right)\\ &= \frac{r+1}{r}\left[\sum_{n=0}^{+\infty}\frac{1}{\binom{n+r}{r}} - \sum_{n=1}^{+\infty}\frac{1}{\binom{n+r}{r}}\right]\\ &= \frac{r+1}{r}\frac{1}{\binom{0+r}{r}}\\ &= \frac{r+1}{r},\end{aligned}$$

assuming the sum converges.

When $r = 2$, we have

$$\sum_{n=1}^{+\infty}\frac{1}{\binom{n+2}{3}} = \frac{3}{2}.$$

When $n = 1$, $\frac{1}{\binom{1+2}{3}} = \frac{1}{1} = 1$.

Therefore,

$$\sum_{n=2}^{+\infty}\frac{1}{\binom{n+2}{3}} = \frac{1}{2}$$

as desired.

2. Notice that

$$\begin{aligned}\frac{3!}{n^3} < \frac{1}{\binom{n+1}{3}} &\iff \frac{3!}{n^3} < \frac{3!}{(n+1)n(n-1)}\\ &\iff n^3 > (n+1)n(n-1)\\ &\iff n^3 > n(n^2-1)\\ &\iff n^3 > n^3 - n\\ &\iff n > 0,\end{aligned}$$

which is true.

Also, notice that

$$\begin{aligned}
\frac{20}{\binom{n+1}{3}} - \frac{1}{\binom{n+2}{5}} < \frac{5!}{n^3} &\iff \frac{5!}{(n+1)(n)(n-1)} - \frac{5!}{(n+2)(n+1)(n)(n-1)(n-2)} < \frac{5!}{n^3} \\
&\iff \frac{(n+2)(n-2)-1}{(n+2)(n+1)(n)(n-1)(n-2)} < \frac{1}{n^3} \\
&\iff (n^2-5)n^3 < (n^2-4)(n^2-1)n \\
&\iff n^5 - 5n^3 < n^5 - 5n^3 + 4n \\
&\iff 4n > 0,
\end{aligned}$$

which is true.

Therefore, we have that

$$\begin{aligned}
\sum_{n=3}^{+\infty} \frac{3!}{n^3} &< \sum_{n=3}^{+\infty} \frac{1}{\binom{n+1}{3}} \\
&= \sum_{n=2}^{+\infty} \frac{1}{\binom{n+2}{3}} \\
&= \frac{1}{2},
\end{aligned}$$

and therefore $\sum_{n=3}^{+\infty} \frac{1}{n^3} < \frac{1}{12}$, and $\sum_{n=1}^{+\infty} \frac{1}{n^3} < 1 + \frac{1}{8} + \frac{1}{12} = \frac{29}{24} = \frac{116}{96}$.

On the other hand, we have

$$\begin{aligned}
\sum_{n=3}^{+\infty} \frac{5!}{n^3} &< \sum_{n=3}^{+\infty} \left[\frac{20}{\binom{n+1}{3}} - \frac{1}{\binom{n+2}{5}} \right] \\
&= 20 \sum_{n=2}^{+\infty} \frac{1}{\binom{n+2}{3}} - \sum_{n=1}^{+\infty} \frac{1}{\binom{n+4}{5}} \\
&= 20 \cdot \frac{1}{2} - \frac{5}{4} \\
&= 10 - \frac{5}{4} \\
&= \frac{35}{4},
\end{aligned}$$

and therefore $\sum_{n=3}^{+\infty} \frac{1}{n^3} > \frac{7}{96}$, and $\sum_{n=1}^{+\infty} \frac{1}{n^3} > 1 + \frac{1}{8} + \frac{7}{96} = \frac{115}{96}$.

Hence,

$$\frac{115}{96} < \sum_{n=1}^{+\infty} \frac{1}{n^3} < \frac{116}{96}$$

as desired.

2017.3 Question 2

1. Let the complex number representing $R(P)$ be z'. Therefore,

$$\begin{aligned} z' - a &= \exp(i\theta)(z-a), \\ z' &= z\exp(i\theta) + a(1-\exp(i\theta)), \end{aligned}$$

as desired.

2. Let the complex number representing $SR(P)$ be z''. Therefore,

$$\begin{aligned} z'' - b &= \exp(i\varphi)(z'-b), \\ z'' &= z'\exp(i\varphi) + b(1-\exp(i\varphi)), \\ z'' &= [z\exp(i\theta) + a(1-\exp(i\theta))]\exp(i\varphi) + b(1-\exp(i\varphi)), \\ z'' &= z\exp(i\,(\theta+\varphi)) + a(1-\exp(i\theta))\exp(i\varphi) + b(1-\exp(i\varphi)). \end{aligned}$$

This will be an anti-clockwise rotation around c over an angle of $(\theta+\varphi)$, where

$$c\,[1-\exp(i(\theta+\varphi))] = a\exp(i\varphi) - a\exp(i\,(\theta+\varphi)) + b - b\exp(i\varphi),$$

If $\theta+\varphi = 2n\pi$ for some integer $n \in \mathbb{Z}$, $1-\exp(i(\theta+\varphi)) = 0$, therefore c cannot be determined. Multiplying both sides by $\exp\left(-\frac{i(\theta+\varphi)}{2}\right)$, we have

$$\begin{aligned} & c\left[\exp\left(-\frac{i(\theta+\varphi)}{2}\right) - \exp\left(\frac{i(\theta+\varphi)}{2}\right)\right] \\ &= a\left[\exp\left(\frac{i(\varphi-\theta)}{2}\right) - \exp\left(\frac{i(\theta+\varphi)}{2}\right)\right] + b\left[\exp\left(-\frac{i(\theta+\varphi)}{2}\right) - \exp\left(\frac{i(\varphi-\theta)}{2}\right)\right], \end{aligned}$$

and hence

$$\begin{aligned} -2ci\sin\left(\frac{\theta+\varphi}{2}\right) &= -2ai\exp\left(\frac{i\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right) - 2bi\exp\left(-\frac{i\theta}{2}\right)\sin\left(\frac{\varphi}{2}\right), \\ c\sin\left(\frac{\theta+\varphi}{2}\right) &= a\exp\left(\frac{i\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right) + b\exp\left(-\frac{i\theta}{2}\right)\sin\left(\frac{\varphi}{2}\right). \end{aligned}$$

If $\theta+\varphi = 2\pi$, we will have $z'' = z + a\exp(i\varphi) - a + b(1-\exp(i\varphi)) = z + (b-a)(1-\exp(i\varphi))$, which is a translation by $(b-a)(1-\exp(i\varphi))$.

3. If $RS = SR$, then we have

$$\begin{aligned} a(1-\exp(i\theta))\exp(i\varphi) + b(1-\exp(i\varphi)) &= b(1-\exp(i\varphi))\exp(i\theta) + a(1-\exp(i\theta)), \\ a(-1+\exp(i\varphi)+\exp(i\theta)-\exp(i(\theta+\varphi))) &= b(-1+\exp(i\varphi)+\exp(i\theta)-\exp(i(\theta+\varphi))), \\ (a-b)(1-\exp(i\varphi))(1-\exp(i\theta)) &= 0. \end{aligned}$$

Therefore, $a = b$, or $\varphi = 2n\pi$, or $\theta = 2n\pi$, for some integer $n \in \mathbb{Z}$.

2017.3 Question 3

By Vieta's Theorem, from the quartic equation in x, we have

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q,$$

and from the cubic equation in y, we have

$$(\alpha\beta + \gamma\delta) + (\alpha\gamma + \beta\delta) + (\alpha\delta + \beta\gamma) = -A.$$

Therefore, $A = -q$.

1. Since $(p, q, r, s) = (0, 3, -6, 10)$, the cubic equation is reduced to

   $$y^3 - 3y^2 - 10y + 84 = 0,$$

   and therefore

   $$(y-2)(y-7)(y+6) = 0.$$

   Therefore, $y_1 = 7, y_2 = 2, y_3 = -6$, and $\alpha\beta + \gamma\delta = 7$.

2. We have

   $$\begin{aligned}(\alpha+\beta)(\gamma+\delta) &= \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta \\ &= (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) - (\alpha\beta + \gamma\delta) \\ &= q - 7 \\ &= 3 - 7 \\ &= -4.\end{aligned}$$

   By Vieta's Theorem, we have $\alpha\beta\gamma\delta = s = 10$. Therefore, $\alpha\beta$ and $\gamma\delta$ must be roots to the equation

   $$x^2 - 7x + 10 = 0.$$

   The two roots are $x = 2$ and $x = 5$, and therefore $\alpha\beta = 5$.

3. We have from the other root that $\gamma\delta = 2$.

   We notice that $(\alpha+\beta) + (\gamma+\delta) = -p = 0$. Therefore, from part 2, $(\alpha+\beta)$ and $(\gamma+\delta)$ are roots to the equation

   $$x^2 - 4 = 0.$$

   This gives us $\alpha + \beta = \pm 2$ and $\gamma + \delta = \mp 2$.

   Using the value of r and Vieta's Theorem, we have

   $$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r = 6.$$

   Plugging in $\alpha\beta = 5$ and $\gamma\delta = 2$, we have

   $$5(\gamma+\delta) + 2(\alpha+\beta) = 6.$$

   Therefore, it must be the case that $\alpha + \beta = -2$ and $\gamma + \delta = 2$.

   Hence, using the values of $\alpha\beta$ and $\gamma\delta$, α and β are solutions to the quadratic equation $x^2+2x+5 = 0$, and γ and δ are solutions to the quadratic equation $x^2 - 2x + 2 = 0$.

   Solving this gives us $\alpha, \beta = -1 \pm 2i$ and $\gamma, \delta = 1 \pm i$. The solutions to the original quartic equation is

   $$x_{1,2} = -1 \pm 2i, x_{3,4} = 1 \pm i.$$

2017.3 Question 4

1. Notice that $a = e^{\ln a}$ and hence $a^x = e^{x \ln a}$, $a^{\frac{x}{\ln a}} = e^x$we have

$$\begin{aligned} F(y) &= \exp\left(\frac{1}{y}\int_0^y \ln f(x)\,\mathrm{d}x\right) \\ &= a^{\frac{1}{y\ln a}\cdot\int_0^y \ln f(x)\,\mathrm{d}x} \\ &= a^{\frac{1}{y}\cdot\int_0^y \frac{\ln f(x)}{\ln a}\,\mathrm{d}x} \\ &= a^{\frac{1}{y}\cdot\int_0^y \log_a f(x)\,\mathrm{d}x} \end{aligned}$$

as desired.

2. We have

$$\begin{aligned} H(y) &= \exp\left(\frac{1}{y}\int_0^y \ln f(x)g(x)\,\mathrm{d}x\right) \\ &= \exp\left[\frac{1}{y}\int_0^y (\ln f(x) + \ln g(x))\,\mathrm{d}x\right] \\ &= \exp\left[\frac{1}{y}\left(\int_0^y \ln f(x)\,\mathrm{d}x + \int_0^y \ln g(x)\,\mathrm{d}x\right)\right] \\ &= \exp\left(\frac{1}{y}\int_0^y \ln f(x)\,\mathrm{d}x\right)\cdot\exp\left(\frac{1}{y}\int_0^y \ln g(x)\,\mathrm{d}x\right) \\ &= F(y)\cdot G(y). \end{aligned}$$

3. Let $f(x) = b^x$.

$$\begin{aligned} F(y) &= \exp\left(\frac{1}{y}\int_0^y \ln f(x)\,\mathrm{d}x\right) \\ &= b^{\frac{1}{y}\int_0^y \log_b f(x)\,\mathrm{d}x} \\ &= b^{\frac{1}{y}\int_0^y \log_b b^x\,\mathrm{d}x} \\ &= b^{\frac{1}{y}\int_0^y x\,\mathrm{d}x} \\ &= b^{\frac{1}{y}\cdot\frac{y^2}{2}} \\ &= b^{\frac{y}{2}} \\ &= \sqrt{b^y}. \end{aligned}$$

4. Since $F(y) = \sqrt{f(y)}$, we notice that $f(y) = F(y)^2 = \exp\left(\frac{2}{y}\int_0^y \ln f(x)\,\mathrm{d}x\right)$, and therefore $\ln f(y) = \frac{2}{y}\int_0^y \ln f(x)\,\mathrm{d}x$.
   We substitute $g(y) = \ln f(y)$, and therefore

$$yg(y) = 2\int_0^y g(y)\,\mathrm{d}x.$$

   Therefore, differentiating both sides with respect to y gives us

$$yg'(y) + g(y) = 2g(y),$$

   and therefore

$$-g(y) + yg'(y) = 0.$$

   Multiplying y^{-2} on both sides gives us

$$-y^{-2}g(y) + y^{-1}g'(y) = 0,$$

   and therefore

$$\frac{\mathrm{d}}{\mathrm{d}y}\frac{g(y)}{y} = 0,$$

and therefore

$$\frac{g(y)}{y} = C \implies g(y) = Cy.$$

Therefore, we have

$$\begin{aligned} f(y) &= \exp g(y) \\ &= \exp(Cy) \\ &= b^y \end{aligned}$$

if we substitute $b = \exp(C) > 0$, and therefore $f(x) = b^y$ as desired.

2017.3 Question 5

Since we have $x = r\cos\theta$ and $y = r\sin\theta$, and $r = f(\theta)$, we have

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}\theta} &= \frac{\mathrm{d}r}{\mathrm{d}\theta}\cdot\cos\theta + r\cdot\frac{\mathrm{d}\cos\theta}{\mathrm{d}\theta}\\ &= f'(\theta)\cos\theta - f(\theta)\sin\theta,\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}\theta} &= \frac{\mathrm{d}r}{\mathrm{d}\theta}\cdot\sin\theta + r\cdot\frac{\mathrm{d}\sin\theta}{\mathrm{d}\theta}\\ &= f'(\theta)\sin\theta + f(\theta)\cos\theta,\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}}\\ &= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}\\ &= \frac{f'(\theta)\tan\theta + f(\theta)}{f'(\theta) - f(\theta)\tan\theta}.\end{aligned}$$

For the two curves, we must have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_f\cdot\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_g = -1$$

for them to meet at right angles. Therefore,

$$\begin{aligned}\frac{f'(\theta)\tan\theta + f(\theta)}{f'(\theta) - f(\theta)\tan\theta}\cdot\frac{g'(\theta)\tan\theta + g(\theta)}{g'(\theta) - g(\theta)\tan\theta} &= -1\\ (f'(\theta)\tan\theta + f(\theta))\cdot(g'(\theta)\tan\theta + g(\theta)) &= -(f'(\theta) - f(\theta)\tan\theta)\cdot(g'(\theta) - g(\theta)\tan\theta)\\ f'(\theta)g'(\theta)(1+\tan^2\theta) + f(\theta)g(\theta)(1+\tan^2\theta) &= 0\\ f'(\theta)g'(\theta) + f(\theta)g(\theta) &= 0.\end{aligned}$$

We have $f\left(-\frac{\pi}{2}\right) = 4$. Let

$$g_a(\theta) = a(1+\sin\theta).$$

Therefore,

$$g_a'(\theta) = a\cos\theta,$$

and we have

$$f'(\theta)(a\cos\theta) + f(\theta)a(1+\sin\theta) = 0,$$

and therefore

$$\frac{\mathrm{d}f(\theta)}{\mathrm{d}\theta}\cos\theta = -f(\theta)(1+\sin\theta).$$

By separating variables we have

$$\frac{\mathrm{d}f(\theta)}{f(\theta)} = -\frac{\mathrm{d}\theta(1+\sin\theta)}{\cos\theta}.$$

Notice that

$$-\frac{1+\sin\theta}{\cos\theta} = -\frac{(1-\sin\theta)(1+\sin\theta)}{(1-\sin\theta)\cos\theta} = -\frac{\cos\theta}{1-\sin\theta} = \frac{\cos\theta}{\sin\theta - 1},$$

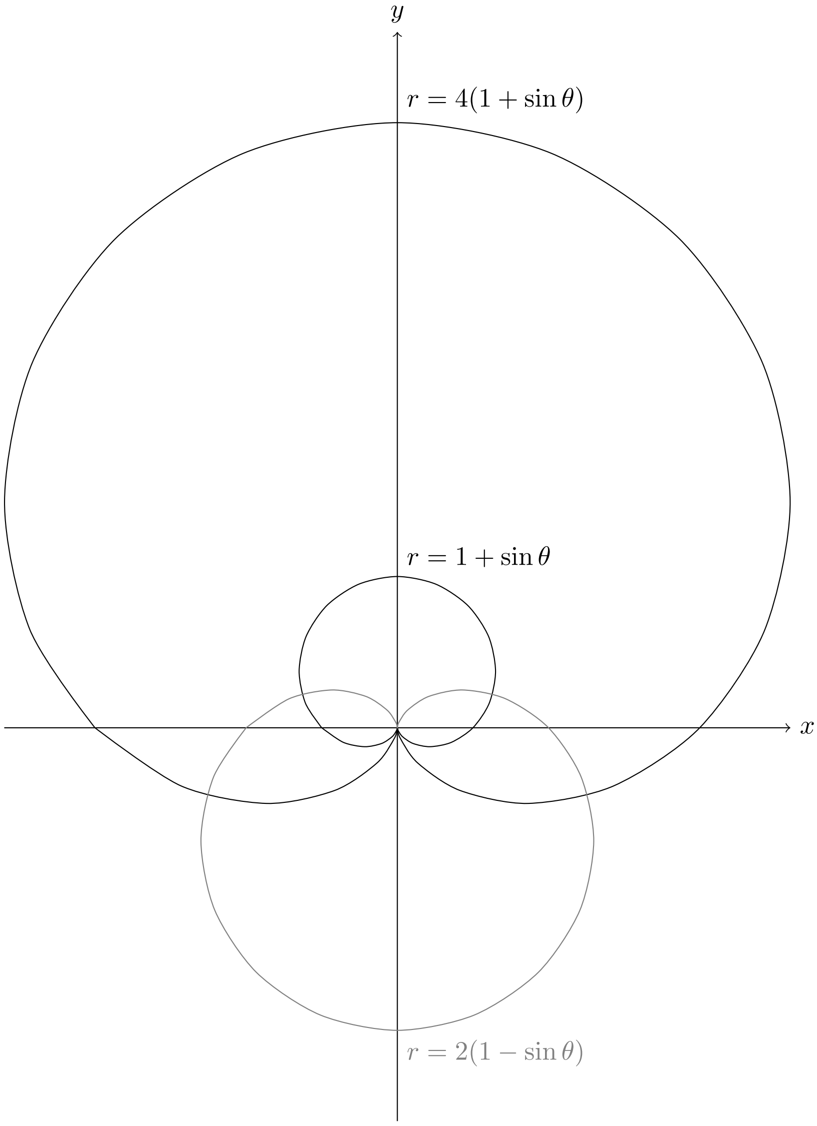
integrating both sides gives us

$$\ln f(\theta) = \ln|\sin\theta - 1| + C = \ln(1-\sin\theta) + C,$$

which gives

$$f(\theta) = A(1-\sin\theta).$$

Since $f\left(-\frac{\pi}{2}\right) = 4$, we have $2A = 4$ and $A = 2$, therefore $f(\theta) = 2(1-\sin\theta)$.



y
$r = 4(1 + \sin\theta)$
$r = 1 + \sin\theta$
x
$r = 2(1 - \sin\theta)$

2017.3 Question 6

1. Consider the substitution $u = \frac{1}{v}$.

   When $u \to 0^+$, $v \to +\infty$.

   When $u = x$, $v = \frac{1}{x}$.

   We also have
   $$\mathrm{d}u = -\frac{1}{v^2}\,\mathrm{d}v.$$

   Therefore,
   $$\begin{aligned}
   T(x) &= \int_0^x \frac{\mathrm{d}u}{1+u^2} \\
   &= \int_{+\infty}^{\frac{1}{x}} -\frac{1}{v^2} \cdot \frac{1}{1+\frac{1}{v^2}}\,\mathrm{d}v \\
   &= \int_{\frac{1}{x}}^{+\infty} \frac{\mathrm{d}v}{1+v^2} \\
   &= \int_0^{+\infty} \frac{\mathrm{d}v}{1+v^2} - \int_0^{\frac{1}{x}} \frac{\mathrm{d}v}{1+v^2} \\
   &= T_\infty - T(x^{-1}),
   \end{aligned}$$

   as desired.

2. When $u \neq a^{-1}$, we have
   $$\begin{aligned}
   \frac{\mathrm{d}v}{\mathrm{d}u} &= \frac{\mathrm{d}}{\mathrm{d}u}\frac{u+a}{1-au} \\
   &= \frac{1 \cdot (1-au) + a \cdot (u+a)}{(1-au)^2} \\
   &= \frac{1-au+au+a^2}{(1-au)^2} \\
   &= \frac{1+a^2}{(1-au)^2}.
   \end{aligned}$$

   Also, notice that
   $$\begin{aligned}
   \frac{1+v^2}{1+u^2} &= \frac{1+\left(\frac{u+a}{1-au}\right)^2}{1+u^2} \\
   &= \frac{(1-au)^2+(u+a)^2}{(1+u^2)(1-au)^2} \\
   &= \frac{1-2au+a^2u^2+u^2+2au+a^2}{(1+u^2)(1-au)^2} \\
   &= \frac{(1+a^2)(1+u^2)}{(1-au)^2(1+u^2)} \\
   &= \frac{1+a^2}{(1-au)^2}.
   \end{aligned}$$

   Therefore, $\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{1+v^2}{1+u^2}$ as desired.

Consider the substitution $v = \frac{u+a}{1-au}$. When $u = 0$, $v = a$. When $u = x, v = \frac{x+a}{1-ax}$. Therefore,

$$\begin{aligned}
T(x) &= \int_0^x \frac{\mathrm{d}u}{1+u^2} \\
&= \int_a^{\frac{x+a}{1-ax}} \frac{1+u^2}{1+v^2} \cdot \frac{\mathrm{d}v}{1+u^2} \\
&= \int_a^{\frac{x+a}{1-ax}} \frac{\mathrm{d}v}{1+v^2} \\
&= \int_0^{\frac{x+a}{1-ax}} \frac{\mathrm{d}v}{1+v^2} - \int_0^a \frac{\mathrm{d}v}{1+v^2} \\
&= T\left(\frac{x+a}{1-ax}\right) - T(a),
\end{aligned}$$

as desired.

If we substitute $T(x) = T_\infty - T(x^{-1})$ and $T(a) = T_\infty - T(a^{-1})$, we can see that

$$\begin{aligned}
T(x) &= T\left(\frac{x+a}{1-ax}\right) - T(a) \\
T_\infty - T(x^{-1}) &= T\left(\frac{x+a}{1-ax}\right) - \left[T_\infty - T(a^{-1})\right] \\
T(x^{-1}) &= 2T_\infty - T\left(\frac{x+a}{1-ax}\right) - T(a^{-1}),
\end{aligned}$$

as desired.

Now, let $y = x^{-1}$ and $b = a^{-1}$. Then

$$\begin{aligned}
\frac{x+a}{1-ax} &= \frac{y^{-1}+b^{-1}}{1-b^{-1}y^{-1}} \\
&= \frac{b+y}{by-1}.
\end{aligned}$$

This gives us

$$T(y) = 2T_\infty - T\left(\frac{b+y}{by-1}\right) - T(b),$$

as desired.

3. Let $y = b = \sqrt{3}$. We can easily verify that $b > 0$ and $y > \frac{1}{b}$. Therefore,

$$T(\sqrt{3}) = 2T_\infty - T\left(\frac{\sqrt{3}+\sqrt{3}}{3-1}\right) - T(\sqrt{3}),$$

which simplified, gives us $T(\sqrt{3}) = \frac{2}{3}T_\infty$ as desired.

In $T(x) = T\left(\frac{x+a}{1-ax}\right) - T(a)$, let $x = a = \sqrt{2}-1$, we can verify that $a > 0$ and $x < \frac{1}{a}$, therefore we have

$$\begin{aligned}
T(\sqrt{2}-1) &= T\left(\frac{(\sqrt{2}-1)+(\sqrt{2}-1)}{1-(\sqrt{2}-1)\cdot(\sqrt{2}-1)}\right) - T(\sqrt{2}-1), \\
T(\sqrt{2}-1) &= T\left(\frac{2\sqrt{2}-2}{1-\left(2+1-2\sqrt{2}\right)}\right) - T(\sqrt{2}-1), \\
T(\sqrt{2}-1) &= T\left(\frac{2\sqrt{2}-2}{2\sqrt{2}-2}\right) - T(\sqrt{2}-1), \\
2T(\sqrt{2}-1) &= T(1).
\end{aligned}$$

In $T(x) = T_\infty - T(x^{-1})$, let $x = 1$. We have

$$T(1) = T_\infty - T(1),$$
$$2T(1) = T_\infty.$$

Therefore, $T(\sqrt{2} - 1) = \frac{1}{4}T_\infty$, as desired.

2017.3 Question 7

$$\begin{aligned}
\frac{x^2}{a^2}+\frac{y^2}{b^2} &= \left(\frac{1-t^2}{1+t^2}\right)^2+\left(\frac{2t}{1+t^2}\right)^2 \\
&= \frac{\left(1-t^2\right)^2+(2t)^2}{\left(1+t^2\right)^2} \\
&= \frac{1-2t^2+t^4+4t^2}{\left(1+t^2\right)^2} \\
&= \frac{1+2t^2+t^4}{\left(1+t^2\right)^2} \\
&= \frac{\left(1+t^2\right)^2}{\left(1+t^2\right)^2} \\
&= 1
\end{aligned}$$

as desired, so T lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

1. The gradient of L must satisfy that

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t} \\
&= \frac{b}{a}\cdot\frac{\mathrm{d}\left(2t/(1+t^2)\right)/\,\mathrm{d}t}{\mathrm{d}\left((1-t^2)/(1+t^2)\right)/\,\mathrm{d}t} \\
&= \frac{b}{a}\cdot\frac{2\cdot(1+t^2)-2t\cdot 2t}{-2t\cdot(1+t^2)-(1-t^2)\cdot 2t} \\
&= \frac{b}{a}\cdot\frac{2+2t^2-4t^2}{-2t-2t^3-2t+2t^3} \\
&= \frac{b}{a}\cdot\frac{1-t^2}{-2t}.
\end{aligned}$$

Therefore, we have a general point $(X,Y)\in L$ satisfy that

$$\begin{aligned}
Y-\frac{2bt}{1+t^2} &= \frac{b}{a}\cdot\frac{1-t^2}{-2t}\cdot\left(X-\frac{a(1-t^2)}{1+t^2}\right) \\
(1+t^2)Y-2bt &= \frac{b}{a}\cdot\frac{1-t^2}{-2t}\cdot\left((1+t^2)X-a(1-t^2)\right) \\
(-2at)(1+t^2)Y-(-2at)(2bt) &= b\cdot(1-t^2)\cdot\left((1+t^2)X-a(1-t^2)\right) \\
(-2at)(1+t^2)Y &= b(1-t^2)(1+t^2)X-ab(1-t^2)^2-4abt^2 \\
(-2at)(1+t^2)Y &= b(1-t^2)(1+t^2)X-ab(1+t^2)^2 \\
-2atY &= b(1-t^2)X-ab(1+t^2) \\
ab(1+t^2)-2atY-b(1-t^2)X &= 0 \\
(a+X)bt^2-2aYt+b(a-X) &= 0
\end{aligned}$$

as desired.

Now if we fix X,Y and solve for t, there are two solutions to this quadratic equation exactly when

$$\begin{aligned}
(2aY)^2-4(a+X)b\cdot b(a-X) &> 0 \\
(aY)^2-(a+X)(a-X)b^2 &> 0 \\
a^2Y^2 &> (a^2-X^2)b^2,
\end{aligned}$$

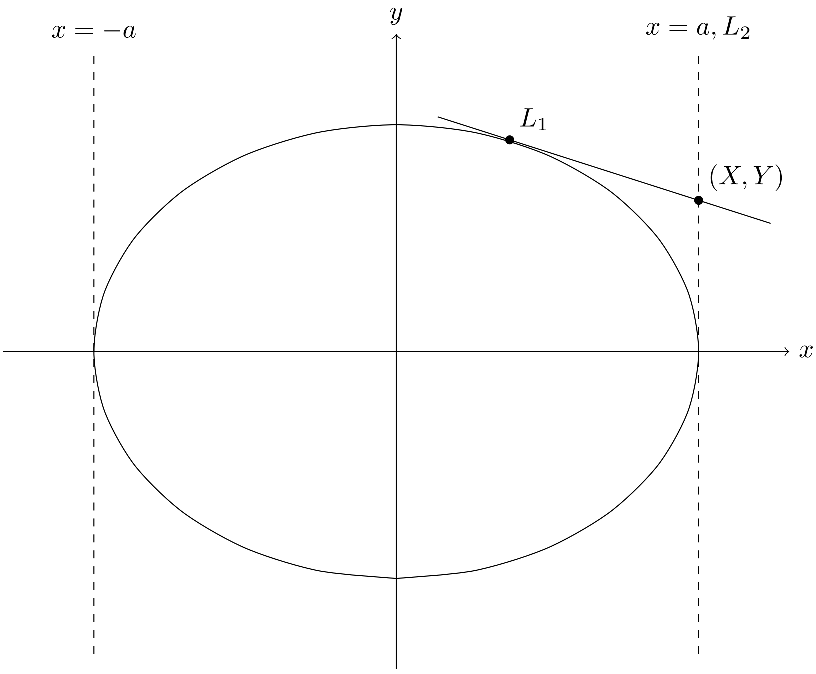
which corresponds to two distinct points on the ellipse.

Since $a^2Y^2 > (a^2 - X^2)b^2$, we have $\frac{Y^2}{b^2} > 1 - \frac{X^2}{a^2}$ by dividing through a^2b^2 on both sides, i.e.

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} > 1,$$

which means when the point (X, Y) lies outside the ellipse.

This also holds when $X^2 = a^2$, i.e. when the point (X, Y) lies on the pair of lines $X = \pm A$. Here, the condition is simply $a^2Y^2 > 0$, which gives $Y \neq 0$. One of the tangents will be the vertical line $X = \pm A$ (whichever one the point lies on), and the other one as a non-vertical (as shown when $X = a$, the tangents being L_1 and L_2).



2. By Vieta's Theorem, we have

$$pq = \frac{b(a - X)}{b(a + X)} \implies (a + X)pq = a - X,$$

as desired, and

$$p + q = -\frac{-2aY}{(a + X)b} = \frac{2aY}{(a + X)b}.$$

Let $X = 0$ for the equation in L,

$$\begin{aligned} abt^2 - 2aYt + ba &= 0 \\ bt^2 - 2Yt + b &= 0 \\ Y &= \frac{b(1 + t^2)}{2t}. \end{aligned}$$

Therefore,

$$\begin{aligned} y_1 + y_2 &= \frac{b(1 + p^2)}{2p} + \frac{b(1 + q^2)}{2q} \\ &= \frac{b\left[(1 + p^2)q + (1 + q^2)p\right]}{2pq} \\ &= 2b, \end{aligned}$$

therefore we have

$$4pq = (1 + p^2)q + (1 + q^2)p = (p + q)(1 + pq).$$

Therefore,

$$
\begin{aligned}
4 \cdot \frac{a-X}{a+X} &= \frac{2aY}{(a+X)b} \cdot \frac{2a}{a+X} \\
a - X &= \frac{a^2Y}{b(a+X)} \\
(a-X)(a+X)b &= a^2Y \\
(a^2 - X^2)b &= a^2Y \\
1 - \frac{X^2}{a^2} &= \frac{Y}{b} \\
\frac{X^2}{a^2} + \frac{Y}{b} &= 1,
\end{aligned}
$$

as desired.

2017.3 Question 8

We have

$$\begin{aligned}\sum_{m=1}^{n} a_m(b_{m+1}-b_m) &= \sum_{m=1}^{n} a_m b_{m+1} - \sum_{m=1}^{n} a_m b_m \\ &= -\sum_{m=0}^{n-1} b_{m+1}a_{m+1} + \sum_{m=1}^{n} b_{m+1}a_m \\ &= -\sum_{m=1}^{n} b_{m+1}a_{m+1} + \sum_{m=1}^{n} b_{m+1}a_m + a_{n+1}b_{n+1} - a_1b_1 \\ &= a_{n+1}b_{n+1} - a_1b_1 - \sum_{m=1}^{n} b_{m+1}(a_{m+1}-a_m),\end{aligned}$$

as desired.

1. Let $a_m = 1$. On one hand, we have

$$\begin{aligned}\sum_{m=1}^{n} a_m(b_{m+1}-b_m) &= \sum_{m=1}^{n} [\sin(m+1)x - \sin mx] \\ &= \sum_{m=1}^{n} 2\cos\left(\frac{(m+1)x+mx}{2}\right)\sin\left(\frac{(m+1)x-mx}{2}\right) \\ &= 2\sum_{m=1}^{n} \cos\left(m+\frac{1}{2}\right)x \sin\frac{x}{2} \\ &= 2\sin\frac{x}{2}\sum_{m=1}^{n} \cos\left(m+\frac{1}{2}\right)x.\end{aligned}$$

   On the other hand, we have

$$\begin{aligned}\sum_{m=1}^{n} a_m(b_{m+1}-b_m) &= a_{n+1}b_{n+1} - a_1b_1 - \sum_{m=1}^{n} b_{m+1}(a_{m+1}-a_m) \\ &= \sin(n+1)x - \sin x.\end{aligned}$$

   Therefore, by rearranging, we have

$$\sum_{m=1}^{n} \cos\left(m+\frac{1}{2}\right)x = \frac{1}{2}\left[\sin(n+1)x - \sin x\right]\operatorname{cosec}\frac{1}{2}x$$

   as desired.

2. Let $a_m = m$, and let $b_m = \cos\left(m-\frac{1}{2}\right)x$. We have the identity

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right).$$

   Therefore, we have

$$\begin{aligned}\sum_{m=1}^{n} a_m(b_{m+1}-b_m) &= \sum_{m=1}^{n} m\cdot\left[\cos\left(m+\frac{1}{2}\right)x - \cos\left(m-\frac{1}{2}\right)x\right] \\ &= \sum_{m=1}^{n} -2m\sin mx\sin\frac{1}{2}x \\ &= -2\sin\frac{1}{2}x\sum_{m=1}^{n} m\sin mx,\end{aligned}$$

and

$$\begin{aligned}
&\sum_{m=1}^{n} a_m(b_{m+1} - b_m) \\
&= a_{n+1}b_{n+1} - a_1 b_1 - \sum_{m=1}^{n} b_{m+1}(a_{m+1} - a_m) \\
&= (n+1)\cos\left(n + \frac{1}{2}\right)x - 1 \cdot \cos\frac{1}{2}x - \sum_{m=1}^{n}\cos\left(m + \frac{1}{2}\right)x \cdot 1 \\
&= (n+1)\cos\left(n + \frac{1}{2}\right)x - \cos\frac{1}{2}x - \sum_{m=1}^{n}\cos\left(m + \frac{1}{2}\right)x \\
&= (n+1)\cos\left(n + \frac{1}{2}\right)x - \cos\frac{1}{2}x - \frac{1}{2}\left(\sin(n+1)x - \sin x\right)\operatorname{cosec}\frac{1}{2}x \\
&= \frac{1}{2}\operatorname{cosec}\frac{1}{2}x\left[2(n+1)\cos\left(n + \frac{1}{2}\right)x\sin\frac{1}{2}x - 2\cos\frac{1}{2}x\sin\frac{1}{2}x - \left(\sin(n+1)x - \sin x\right)\right] \\
&= \frac{1}{2}\operatorname{cosec}\frac{1}{2}x\left[(n+1)\left(\sin(n+1)x - \sin nx\right) - (\sin x - \sin 0) - \left(\sin(n+1)x - \sin x\right)\right] \\
&= \frac{1}{2}\operatorname{cosec}\frac{1}{2}x\left[n\sin(n+1)x - (n+1)\sin nx\right].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
-2\sin\frac{1}{2}x\sum_{m=1}^{n} m\sin mx &= \frac{1}{2}\operatorname{cosec}\frac{1}{2}x\left[n\sin(n+1)x - (n+1)\sin nx\right] \\
\sum_{m=1}^{n} m\sin mx &= -\frac{1}{4}\operatorname{cosec}^2\frac{1}{2}x\left[n\sin(n+1)x - (n+1)\sin nx\right],
\end{aligned}$$

and therefore, $p = -\frac{1}{4}n$, $q = \frac{1}{4}(n+1)$.

2017.3 Question 12

1. First, note that

$$\begin{aligned}1 &= \sum_{x,y=1}^{x=n} \mathrm{P}(X=x, Y=y)\\ &= \sum_{x=1}^{n}\sum_{y=1}^{n} k(x+y)\\ &= \sum_{x=1}^{n}\sum_{y=1}^{n}(kx+ky)\\ &= \sum_{x=1}^{n}\left(n\cdot kx + k\sum_{y=1}^{n} y\right)\\ &= nk\sum_{x=1}^{n} x + nk\sum_{y=1}^{n} y\\ &= n^2(n+1)k\end{aligned}$$

   Therefore, $k = \frac{1}{n^2(n+1)}$

$$\begin{aligned}\mathrm{P}(X=x) &= \sum_{y=1}^{n} \mathrm{P}(X=x, Y=y)\\ &= \sum_{y=1}^{n} k(x+y)\\ &= nkx + k\sum_{y=1}^{n} y\\ &= nkx + \frac{kn(n+1)}{2}\\ &= \frac{x}{n(n+1)} + \frac{1}{2n}\\ &= \frac{2x+n+1}{2n(n+1)},\end{aligned}$$

   as desired.
   By symmetry, $\mathrm{P}(Y=y) = \frac{2y+n+1}{2n(n+1)}$.
   We have

$$\mathrm{P}(X=x)\cdot\mathrm{P}(Y=y) = \frac{(2x+n+1)(2y+n+1)}{4n^2(n+1)^2}.$$

   But $\mathrm{P}(X=x, Y=y) = \frac{x+y}{n^2(n+1)}$ is not equal to this. So X and Y are not independent.

2. By definition,

$$\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y).$$

We have

$$\begin{aligned}
\mathrm{E}(X) = \mathrm{E}(Y) &= \sum_{t=1}^{n} t \cdot \mathrm{P}(X = t) \\
&= \sum_{t=1}^{n} \frac{t \cdot (2t + n + 1)}{2n(n+1)} \\
&= \frac{1}{n(n+1)} \sum_{t=1}^{n} t^2 + \frac{1}{2n} \sum_{t=1}^{n} t \\
&= \frac{n(n+1)(2n+1)}{6n(n+1)} + \frac{n(n+1)}{4n} \\
&= \frac{2n+1}{6} + \frac{n+1}{4} \\
&= \frac{4n + 2 + 3n + 3}{12} \\
&= \frac{7n+5}{12},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}(XY) &= \sum_{x,y=1}^{n} xy \cdot \mathrm{P}(X = x, Y = y) \\
&= \sum_{x=1}^{n} \sum_{y=1}^{n} \frac{xy(x+y)}{n^2(n+1)} \\
&= \frac{1}{n^2(n+1)} \sum_{x=1}^{n} \sum_{y=1}^{n} xy(x+y) \\
&= \frac{1}{n^2(n+1)} \sum_{x=1}^{n} \sum_{y=1}^{n} (x^2 y + xy^2) \\
&= \frac{1}{n^2(n+1)} \left[\sum_{x=1}^{n} x^2 \sum_{y=1}^{n} y + \sum_{x=1}^{n} x \sum_{y=1}^{n} y^2 \right] \\
&= \frac{1}{n^2(n+1)} \cdot 2 \cdot \frac{n(n+1)(2n+1)}{6} \cdot \frac{n(n+1)}{2} \\
&= \frac{(2n+1)(n+1)}{6}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Cov}(X, Y) &= \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y) \\
&= \frac{(2n+1)(n+1)}{6} - \frac{(7n+5)^2}{144} \\
&= \frac{48n^2 + 72n + 24}{144} - \frac{49n^2 + 70n + 25}{144} \\
&= \frac{-n^2 + 2n - 1}{144} \\
&= -\frac{(n-1)^2}{144} \\
&< 0,
\end{aligned}$$

as desired.

2017.3 Question 13

We have

$$\begin{aligned}V(x) &= \mathrm{E}[(X-x)^2] \\ &= \mathrm{E}(X^2 - 2xX + x^2) \\ &= \mathrm{E}(X^2) - 2x\,\mathrm{E}(X) + x^2 \\ &= \sigma^2 + \mu^2 - 2x\mu + x^2.\end{aligned}$$

Therefore, if $Y = V(X)$, then

$$\begin{aligned}\mathrm{E}(Y) &= \mathrm{E}(V(X)) \\ &= \mathrm{E}(\sigma^2 + \mu^2 - 2X\mu + X^2) \\ &= \sigma^2 + \mu^2 - 2\mu\,\mathrm{E}(X) + \mathrm{E}(X^2) \\ &= \sigma^2 + \mu^2 - 2\mu^2 + \mu^2 + \sigma^2 \\ &= 2\sigma^2.\end{aligned}$$

Let $X \sim U[0,1]$, we have $\mu = \mathrm{E}(X) = \frac{1}{2}$, and $\sigma^2 = \mathrm{Var}(X) = \frac{1}{12}$. Therefore,

$$\begin{aligned}V(x) &= \frac{1}{12} + \frac{1}{4} - x + x^2 \\ &= x^2 - x + \frac{1}{3}.\end{aligned}$$

The c.d.f. of X is F, defined as

$$\mathrm{P}(X \leq x) = F(x) = \begin{cases} 0, & x \leq 0, \\ x, & 0 < x \leq 1, \\ 1, & 1 < x \end{cases}$$

Let the c.d.f. of Y be G, we have $G(y) = \mathrm{P}(Y \leq y)$.
Since $V([0,1]) = \left[\frac{1}{12}, \frac{1}{3}\right]$, we must have $G(y) = 0$ for $y \leq \frac{1}{12}$ and $G(y) = 1$ for $y > \frac{1}{3}$.
For $y \in \left(\frac{1}{12}, \frac{1}{3}\right]$, we have

$$\begin{aligned}G(y) = \mathrm{P}(Y \leq y) &= \mathrm{P}(V(X) \leq y) \\ &= \mathrm{P}\left(\left(x - \frac{1}{2}\right)^2 + \frac{1}{12} \leq y\right) \\ &= \mathrm{P}\left(\left|x - \frac{1}{2}\right| \leq \sqrt{y - \frac{1}{12}}\right) \\ &= \mathrm{P}\left(\frac{1}{2} - \sqrt{y - \frac{1}{12}} \leq x \leq \frac{1}{2} + \sqrt{y - \frac{1}{12}}\right) \\ &= F\left(\frac{1}{2} + \sqrt{y - \frac{1}{12}}\right) - F\left(\frac{1}{2} - \sqrt{y - \frac{1}{12}}\right) \\ &= \left(\frac{1}{2} + \sqrt{y - \frac{1}{12}}\right) - \left(\frac{1}{2} - \sqrt{y - \frac{1}{12}}\right) \\ &= 2\sqrt{y - \frac{1}{12}}.\end{aligned}$$

Therefore, the p.d.f. of y, g satisfies that for $y \in \left(\frac{1}{12}, \frac{1}{3}\right]$,

$$g(y) = G'(y) = \frac{1}{\sqrt{y - \frac{1}{12}}}$$

and 0 everywhere else.

Hence, we have

$$\begin{aligned}
\mathrm{E}(Y) &= \int_{\mathbb{R}} y f(y) \, \mathrm{d}y \\
&= \int_{\frac{1}{12}}^{\frac{1}{3}} \frac{y}{\sqrt{y - \frac{1}{12}}} \, \mathrm{d}y \\
&= \int_{y=\frac{1}{12}}^{y=\frac{1}{3}} 2y \, \mathrm{d}\sqrt{y - \frac{1}{12}} \\
&= \left[2y\sqrt{y - \frac{1}{12}} \right]_{\frac{1}{12}}^{\frac{1}{3}} - 2\int_{\frac{1}{12}}^{\frac{1}{3}} \sqrt{y - \frac{1}{12}} \, \mathrm{d}y \\
&= \left[2y\sqrt{y - \frac{1}{12}} - \frac{4}{3}\left(y - \frac{1}{12}\right)^{\frac{3}{2}} \right]_{\frac{1}{12}}^{\frac{1}{3}} \\
&= 2 \cdot \frac{1}{3} \cdot \frac{1}{2} - \frac{4}{3} \cdot \frac{1}{8} \\
&= \frac{1}{6}.
\end{aligned}$$

Also, $2\sigma^2 = 2 \cdot \frac{1}{12} = \frac{1}{6} = \mathrm{E}(Y)$, so the formula we derived holds in this case.