Year 2024

2024 Paper 2

2024.2 Question 1

1. In the $n+k$ integers, the first one is c, and the final one is $c+n+k-1$.

   In the n integers, the first one is $c+n+k$, and the final one is $c+2n+k-1$.

   Hence, the sums are equal if and only if

$$\begin{aligned}\frac{(n+k)[c+(c+n+k-1)]}{2} &= \frac{n[(c+n+k)+(c+2n+k-1)]}{2}\\ (n+k)(2c+n+k-1) &= n(2c+3n+2k-1)\\ 2cn+n^2+nk-n+2ck+kn+k^2-k &= 2cn+3n^2+2kn-1\\ 2ck+k^2 &= 2n^2+k,\end{aligned}$$

   as desired. All the above steps are reversible.

2. (a) When $k=1$, $2c+1=2n^2+1$, and $c=n^2$.
   Hence,
$$(c,n)\in\left\{(t^2,t)\mid t\in\mathbb{N}\right\},$$
   and n can take all positive integers.

   (b) When $k=2$, $4c+4=2n^2+2$, and $2c=n^2-1$.
   By parity, n must be odd. Let $n=2t-1$ for $t\in\mathbb{N}$, and we have
$$2c=(2t-1)^2-1=4t^2-4t,$$
   and hence
$$c=2t^2-2t.$$
   Hence,
$$(c,n)\in\left\{(2t^2-2t,2t-1)\mid t\in\mathbb{N}\right\},$$
   and n can take all odd positive integers.

3. If $k=4$, we have $8c+16=2n^2+4$, and hence $n^2=4c+6$.

   By considering modulo 4, the only quadratic residues modulo 4 are 0 and 1, but the right-hand side equation is congruent to 2 modulo 4.

   Hence, there are no solutions for n and c.

4. When $c=1$, we have $2n^2+k=2k+k^2$, and hence $2n^2=k^2+k$.

   (a) When $k=1$, $k^2+k=2$, and so $(n,k)=(1,1)$ satisfies the equation.
   When $k=8$, $k^2+k=64+8=72$, and so $(n,k)=(6,8)$ satisfies the equation.

   (b) Given that $2N^2=K^2+K$, notice that

$$\begin{aligned}(2N'^2)-(K'^2+K') &= 2(3N+2K+1)^2-(4N+3K+1)^2-(4N+3K+1)\\ &= 2(9N^2+4K^2+1+12NK+6N+4K)\\ &\quad-(16N^2+9K^2+1+24NK+8N+6K)\\ &\quad-(4N+3K+1)\\ &= 2N^2-K^2-K\\ &= 2N^2-(K^2+K)\\ &= 2N^2-2N^2\\ &= 0,\end{aligned}$$

   and this means that
$$2N'^2=K'^2+K',$$
   and hence
$$(N',K')=(3N+2K+1,4N+3K+1)$$
   is another pair of solution for (n,k).

   (c) When $(n,k)=(6,8)$, $3n+2k+1=35$, $4n+3k+1=49$, and $(n,k)=(35,49)$ is also possible.
   When $(n,k)=(35,49)$, $3n+2k+1=204, 4n+3k+1=288$, and $(n,k)=(204,288)$ is also possible.

2024.2 Question 2

1. By Newton's binomial theorem, we have

$$\begin{aligned}(8+x^3)^{-1} &= \frac{1}{8}\left(1+\left(\frac{x}{2}\right)^3\right)^{-1} \\ &= \frac{1}{8}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^{3k},\end{aligned}$$

and this is valid for

$$\left|\frac{x}{2}\right| < 1, |x| < 2,$$

as desired.

Hence,

$$\begin{aligned}\int_0^1 \frac{x^m}{8+x^3}\,\mathrm{d}x &= \int_0^1 \frac{1}{8}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^{3k} x^m\,\mathrm{d}x \\ &= \frac{1}{8}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3k}}\int_0^1 x^{3k+m}\,\mathrm{d}x \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\left[\frac{x^{3k+m+1}}{3k+m+1}\right]_0^1 \\ &= \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2^{3(k+1)}}\cdot\frac{1}{3k+m+1}\right),\end{aligned}$$

as desired.

2. Let $m = 2$, and we have

$$\int_0^1 \frac{x^2}{8+x^3}\,\mathrm{d}x = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2^{3(k+1)}}\cdot\frac{1}{3k+3}\right).$$

Let $m = 1$, and we have

$$\int_0^1 \frac{x}{8+x^3}\,\mathrm{d}x = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2^{3(k+1)}}\cdot\frac{1}{3k+2}\right).$$

Let $m = 0$, and we have

$$\int_0^1 \frac{x}{8+x^3}\,\mathrm{d}x = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2^{3(k+1)}}\cdot\frac{1}{3k+1}\right).$$

Hence,

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\left(\frac{1}{3k+3}-\frac{2}{3k+2}+\frac{4}{3k+1}\right) &= \int_0^1 \frac{x^2}{8+x^3}\,\mathrm{d}x - 2\int_0^1 \frac{x}{8+x^3}\,\mathrm{d}x + 4\int_0^1 \frac{\mathrm{d}x}{8+x^3} \\ &= \int_0^1 \frac{x^2-2x+4}{8+x^3}\,\mathrm{d}x \\ &= \int_0^1 \frac{x^2-2x+4}{(x+2)(x^2-2x+4)}\,\mathrm{d}x \\ &= \int_0^1 \frac{\mathrm{d}x}{x+2} \\ &= [\ln|x+2|]_0^1 \\ &= \ln 3 - \ln 2.\end{aligned}$$

3. Using partial fractions, let A' and B' be real constants such that

$$\begin{aligned}\frac{72(2k+1)}{(3k+1)(3k+2)} &= \frac{A'}{3k+1} + \frac{B'}{3k+2} \\ &= \frac{3(A'+B')k + (2A'+B')}{(3k+1)(3k+2)}.\end{aligned}$$

Hence, we have

$$\begin{cases} 3(A'+B') = 72 \cdot 2 = 144, \\ 2A'+B' = 72. \end{cases}$$

Therefore, $(A', B') = (24, 24)$.

Let

$$A = \int_0^1 \frac{\mathrm{d}x}{8+x^3}, B = \int_0^1 \frac{x\,\mathrm{d}x}{8+x^3}, C = \int_0^1 \frac{x^2\,\mathrm{d}x}{8+x^3},$$

and what is desired is $24(A+B)$.

From the previous part, we can see that $4A - 2B + C = \ln 3 - \ln 2$.

Also,

$$\begin{aligned}2A + B &= \int_0^1 \frac{(2+x)\,\mathrm{d}x}{8+x^3} \\ &= \int_0^1 \frac{\mathrm{d}x}{x^2-2x+4} \\ &= \int_0^1 \frac{\mathrm{d}x}{(x-1)^2+3} \\ &= \frac{1}{\sqrt{3}}\left[\arctan\left(\frac{x-1}{\sqrt{3}}\right)\right]_0^1 \\ &= \frac{1}{\sqrt{3}} \cdot \left[\arctan 0 - \arctan\left(-\frac{1}{\sqrt{3}}\right)\right] \\ &= \frac{1}{\sqrt{3}} \cdot \frac{\pi}{6} \\ &= \frac{\pi}{6\sqrt{3}}.\end{aligned}$$

We also have

$$\begin{aligned}C &= \int_0^1 \frac{x^2\,\mathrm{d}x}{8+x^3} \\ &= \frac{1}{3}\left[\ln(8+x^3)\right]_0^1 \\ &= \frac{1}{3}\left[\ln 9 - \ln 8\right] \\ &= \frac{2}{3}\ln 3 - \ln 2.\end{aligned}$$

Hence, we have

$$4A - 2B = \ln 3 - \ln 2 - \frac{2}{3}\ln 3 + \ln 2 = \frac{1}{3}\ln 3,$$

and hence $2A - B = \frac{1}{6}\ln 3$.

Therefore,

$$4A = \frac{1}{6}\ln 3 + \frac{\pi}{6\sqrt{3}},$$

and hence

$$A = \frac{\ln 3}{24} + \frac{\pi}{24\sqrt{3}}.$$

Subtracting two of this from $2A + B$ gives

$$B = \frac{\pi}{6\sqrt{3}} - \frac{\ln 3}{12} - \frac{\pi}{12\sqrt{3}} = \frac{\pi}{12\sqrt{3}} - \frac{\ln 3}{12},$$

and hence what is desired is

$$\begin{aligned}24(A + B) &= 24\left(\frac{\pi}{24\sqrt{3}} + \frac{\pi}{12\sqrt{3}} + \frac{\ln 3}{24} - \frac{\ln 3}{12}\right) \\ &= 24\left(\frac{\pi}{8\sqrt{3}} - \frac{\ln 3}{24}\right) \\ &= \pi\sqrt{3} - \ln 3,\end{aligned}$$

which gives $a = 3, b = 3$.

2024.2 Question 3

1. The line NP has gradient
$$m_{NP} = \frac{\sin\theta - 0}{\cos\theta - (-1)} = \frac{\sin\theta}{\cos\theta + 1},$$
and hence it has equation
$$l_{NP} : y = \frac{\sin\theta}{\cos\theta + 1} \cdot (x+1).$$
When $x = 0$, we have
$$\begin{aligned} q &= \frac{\sin\theta}{\cos\theta + 1} \\ &= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} - 1 + 1} \\ &= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \\ &= \tan\frac{\theta}{2}. \end{aligned}$$

2. (a)
$$\begin{aligned} \text{RHS} &= \tan\frac{1}{2}\left(\theta + \frac{1}{2}\pi\right) \\ &= \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) \\ &= \frac{\tan\frac{\theta}{2} + \tan\frac{\pi}{4}}{1 - \tan\frac{\theta}{2}\tan\frac{\pi}{4}} \\ &= \frac{q+1}{1-q} \\ &= f_1(q), \end{aligned}$$
as desired.

   (b) Let the coordinates of P_1 be $(\cos\varphi, \sin\varphi)$, and hence we must have
$$\begin{aligned} f_1(q) &= \tan\frac{1}{2}\varphi \\ \tan\frac{1}{2}\left(\theta + \frac{1}{2}\pi\right) &= \tan\frac{1}{2}\varphi \\ \varphi &= \theta + \frac{1}{2}\pi, \end{aligned}$$
and so P_1 is the image of P being rotated through an angle of π counterclockwise about the origin.

3. (a) The coordinates of P_2 are $\left(\cos\left(\theta + \frac{1}{3}\pi\right), \sin\left(\theta + \frac{1}{3}\pi\right)\right)$, and hence we must have that
$$\begin{aligned} f_3(q) &= \tan\frac{1}{2}\left(\theta + \frac{1}{3}\pi\right) \\ &= \tan\left(\frac{\theta}{2} + \frac{\pi}{6}\right) \\ &= \frac{\tan\frac{\theta}{2} + \tan\frac{\pi}{6}}{1 - \tan\frac{\theta}{2}\tan\frac{\pi}{6}} \\ &= \frac{q + \frac{1}{\sqrt{3}}}{1 - q \cdot \frac{1}{\sqrt{3}}} \\ &= \frac{1 + \sqrt{3}q}{\sqrt{3} - q}. \end{aligned}$$

(b) Notice that $f_3(q) = f_1(-q) = \tan\frac{1}{2}\left(-\theta + \frac{1}{2}\pi\right)$, and so the coordinates of P_3 must be

$$\left(\cos\left(\frac{1}{2}\pi - \theta\right), \sin\left(\frac{1}{2}\pi - \theta\right)\right),$$

which is $P_3(\sin\theta, \cos\theta)$, a reflection of P in the line $y = x$.

(c) P_4 must be the image of P under the following transformations:

- Rotation counterclockwise by $\frac{1}{3}\pi$ about the origin O;
- Reflection in the line $y = x$;
- Rotation clockwise by $\frac{1}{3}\pi$ about the origin O.

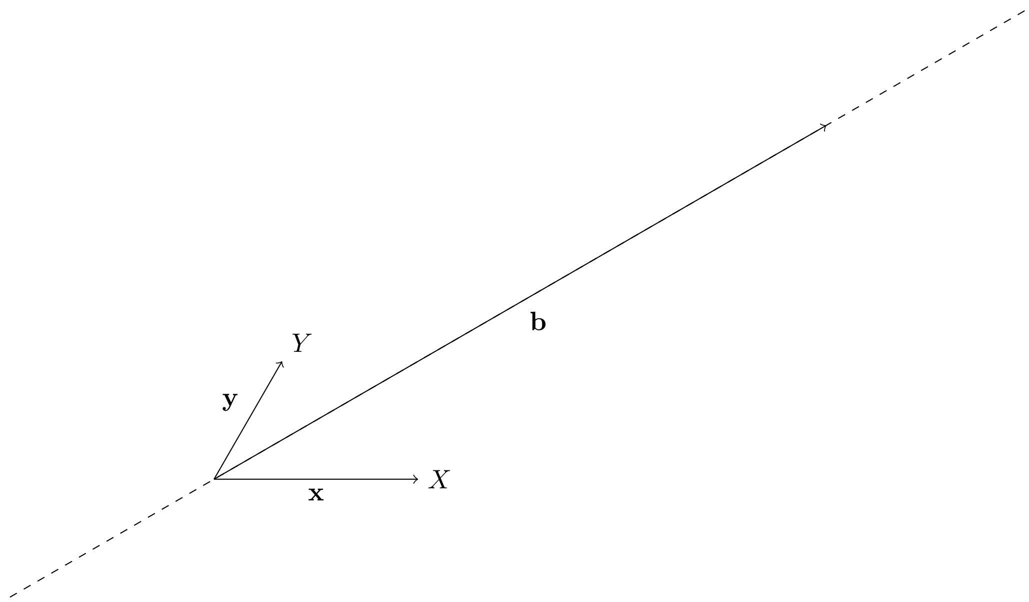
This is precisely the reflection in which the axis after the second step is $y = x$. Hence, the axis of this reflection has an angle of $\frac{1}{4}\pi - \frac{1}{3}\pi = \frac{1}{12}\pi$ with the positive x-axis.

P_4 is the image of P reflected in the line which makes an angle of $-\frac{\pi}{12}$ with the positive x-axis, passing through the origin.

2024.2 Question 4

1. (a) We first show that $\mathbf{b}$ lies in the plane XOY. Since $\mathbf{b}$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$, it must lie in the plane containing $\mathbf{x} = \overrightarrow{OX}$ and $\mathbf{y} = \overrightarrow{OY}$, which is the plane XOY.
   Let α be the angle between $\mathbf{b}$ and $\mathbf{x}$, and let β be the angle between $\mathbf{b}$ and $\mathbf{y}$, where $0 \leq \alpha, \beta \leq \pi$.
   We have
   $$\begin{aligned}\cos\alpha &= \frac{\mathbf{b}\cdot\mathbf{x}}{|\mathbf{b}||\mathbf{x}|} \\ &= \frac{1}{|\mathbf{b}|}\cdot\frac{(|\mathbf{x}|\mathbf{y}+|\mathbf{y}|\mathbf{x})\cdot\mathbf{x}}{|\mathbf{x}|} \\ &= \frac{1}{|\mathbf{b}|}\cdot\frac{|\mathbf{x}|\cdot(\mathbf{x}\cdot\mathbf{y})+|\mathbf{y}|\cdot|\mathbf{x}|^2}{|\mathbf{x}|} \\ &= \frac{1}{|\mathbf{b}|}\cdot(\mathbf{x}\cdot\mathbf{y}+|\mathbf{x}|\cdot|\mathbf{y}|).\end{aligned}$$
   Similarly,
   $$\cos\beta = \frac{1}{|\mathbf{b}|}\cdot(\mathbf{x}\cdot\mathbf{y}+|\mathbf{x}|\cdot|\mathbf{y}|) = \cos\alpha.$$
   Since the cos function is one-to-one on $[0, \pi]$, we must have $\alpha = \beta$.
   Since $\mathbf{x}\cdot\mathbf{y} = |\mathbf{x}|\cdot|\mathbf{y}|\cdot\cos\theta$ where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$, we have $\mathbf{x}\cdot\mathbf{y} \geq -|\mathbf{x}||\mathbf{y}|$, and since $\theta \neq \pi$ (since OXY are non-collinear), we have $\mathbf{x}\cdot\mathbf{y} > -|\mathbf{x}||\mathbf{y}|$, and hence $\cos\alpha = \cos\beta > 0$.
   This shows that both angles are less than $\frac{\pi}{2} = 90°$.
   Hence, the three conditions
   - $\mathbf{b}$ lies in the plane OXY,
   - the angle between $\mathbf{b}$ and $\mathbf{x}$ is equal to the angle between $\mathbf{b}$ and $\mathbf{y}$,
   - both angles are less than $\frac{\pi}{2} = 90°$

   are all satisfied, and we can conclude that $\mathbf{b}$ is a bisecting vector for the plane OXY.



   All bisecting vectors must lie on the line containing $\mathbf{b}$ (the dashed line on the diagram), and hence a scalar multiple of $\mathbf{b}$.
   Furthermore, since both angles must be less than $\frac{\pi}{2}$, it must not on the opposite as where $\mathbf{b}$ is situated, and hence it must be a positive multiple of $\mathbf{b}$.

   (b) If B lies on XY, then $\mathbf{OB} = \mu\mathbf{x} + (1-\mu)\mathbf{y}$ must be a convex combination of $\mathbf{x}$ and $\mathbf{y}$, and hence
   $$\lambda(|\mathbf{x}|\mathbf{y}+|\mathbf{y}|\mathbf{x}) = \mu\mathbf{x}+(1-\mu)\mathbf{y}.$$

Since O, X and Y are not collinear, we must have $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, and hence $\lambda|\mathbf{y}| = \mu$ and $\lambda|\mathbf{x}| = 1 - \mu$, hence giving

$$\lambda = \frac{1}{|\mathbf{x}| + |\mathbf{y}|}$$

We therefore have

$$\begin{aligned}
\frac{XB}{BY} &= \frac{\left|\overrightarrow{OB} - \mathbf{x}\right|}{\left|\mathbf{y} - \overrightarrow{OB}\right|} \\
&= \frac{\left|\frac{|\mathbf{x}|}{|\mathbf{x}|+|\mathbf{y}|}\mathbf{y} + \frac{|\mathbf{y}|}{|\mathbf{x}|+|\mathbf{y}|}\mathbf{y}\mathbf{x} - \mathbf{x}\right|}{\left|\frac{|\mathbf{x}|}{|\mathbf{x}|+|\mathbf{y}|}\mathbf{y} + \frac{|\mathbf{y}|}{|\mathbf{x}|+|\mathbf{y}|}\mathbf{y}\mathbf{x} - \mathbf{y}\right|} \\
&= \frac{\left|\frac{|\mathbf{x}|}{|\mathbf{x}|+|\mathbf{y}|}(\mathbf{y} - \mathbf{x})\right|}{\left|\frac{|\mathbf{y}|}{|\mathbf{x}|+|\mathbf{y}|}(\mathbf{x} - \mathbf{y})\right|} \\
&= \frac{\frac{|\mathbf{x}|}{|\mathbf{x}|+|\mathbf{y}|} \cdot |\mathbf{y} - \mathbf{x}|}{\frac{|\mathbf{y}|}{|\mathbf{x}|+|\mathbf{y}|} \cdot |\mathbf{x} - \mathbf{y}|} \\
&= \frac{|\mathbf{x}|}{|\mathbf{y}|},
\end{aligned}$$

which means

$$XB : BY = |\mathbf{x}| : |\mathbf{y}|,$$

which is precisely the angle bisector theorem.

(c) Considering the dot product,

$$\begin{aligned}
\overrightarrow{OB} \cdot \overrightarrow{XY} &= \lambda\mathbf{b} \cdot (\mathbf{y} - \mathbf{x}) \\
&= \lambda\left(|\mathbf{x}|\mathbf{y} + |\mathbf{y}|\mathbf{x}\right) \cdot (\mathbf{y} - \mathbf{x}) \\
&= \lambda\left[|\mathbf{x}| \cdot \mathbf{y} \cdot \mathbf{y} + |\mathbf{y}| \cdot \mathbf{x} \cdot \mathbf{y} - |\mathbf{x}| \cdot \mathbf{x} \cdot \mathbf{y} - |\mathbf{y}| \cdot \mathbf{x} \cdot \mathbf{x}\right] \\
&= \lambda\left[|\mathbf{x}| \cdot |\mathbf{y}|^2 + \left[|\mathbf{y}| - |\mathbf{x}|\right]\mathbf{x} \cdot \mathbf{y} - |\mathbf{y}| \cdot |\mathbf{x}|^2\right] \\
&= \lambda\left(|\mathbf{y}| - |\mathbf{x}|\right)\left(|\mathbf{x}||\mathbf{y}| + \mathbf{x} \cdot \mathbf{y}\right) \\
&= 0.
\end{aligned}$$

Since O, X, Y are not collinear, $\mathbf{x} \cdot \mathbf{y} > -|\mathbf{x}||\mathbf{y}|$, and hence $|\mathbf{x}||\mathbf{y}| + \mathbf{x} \cdot \mathbf{y} > 0$.
Also, $\lambda = \frac{1}{|\mathbf{x}|+|\mathbf{y}|} \neq 0$.
So it must be the case that $|\mathbf{x}| - |\mathbf{y}| = 0$, which means $|\mathbf{x}| = |\mathbf{y}|$.
Hence, $OX = OY$, and triangle OXY is isosceles.

2. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be the bisecting vectors for QOR, ROP and POQ respectively, and let $\mathbf{p} = \overrightarrow{OP}$, $\mathbf{q} = \overrightarrow{OQ}$, $\mathbf{r} = \overrightarrow{OR}$.

   Let i, j, k be some arbitrary positive real constant.

   From the question, we have

$$\begin{cases} \mathbf{u} = i\left(|\mathbf{q}|\mathbf{r} + |\mathbf{r}|\mathbf{q}\right), \\ \mathbf{v} = j\left(|\mathbf{r}|\mathbf{p} + |\mathbf{p}|\mathbf{r}\right), \\ \mathbf{w} = k\left(|\mathbf{p}|\mathbf{q} + |\mathbf{q}|\mathbf{p}\right). \end{cases}$$

   Considering a pair of dot-product, we have

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= ij \cdot \left(|\mathbf{q}||\mathbf{r}|\mathbf{r} \cdot \mathbf{p} + |\mathbf{p}||\mathbf{q}|\mathbf{r} \cdot \mathbf{r} + |\mathbf{r}||\mathbf{r}|\mathbf{p} \cdot \mathbf{q} + |\mathbf{r}||\mathbf{p}|\mathbf{q} \cdot \mathbf{r}\right) \\
&= ij|\mathbf{r}|\left(|\mathbf{q}|\mathbf{r} \cdot \mathbf{q} + |\mathbf{p}|\mathbf{r} \cdot \mathbf{q} + |\mathbf{p}||\mathbf{q}||\mathbf{r}| + |\mathbf{r}|\mathbf{p} \cdot \mathbf{q}\right) \\
&= ij|\mathbf{r}|^2|\mathbf{p}||\mathbf{q}|\left(\cos\langle\mathbf{p}, \mathbf{r}\rangle + \cos\langle\mathbf{r}, \mathbf{q}\rangle + \cos\langle\mathbf{p}, \mathbf{q}\rangle + 1\right),
\end{aligned}$$

where $\langle \mathbf{a}, \mathbf{b} \rangle$ denotes the angle between $\mathbf{a}$ and $\mathbf{b}$, in $[0, \pi]$.

Denote

$$t = \cos\langle \mathbf{p}, \mathbf{r} \rangle + \cos\langle \mathbf{r}, \mathbf{q} \rangle + \cos\langle \mathbf{q}, \mathbf{p} \rangle + 1,$$

and hence

$$\begin{cases} \mathbf{u} \cdot \mathbf{v} = ij|\mathbf{r}|^2|\mathbf{p}||\mathbf{q}|t, \\ \mathbf{u} \cdot \mathbf{w} = ik|\mathbf{r}||\mathbf{p}||\mathbf{q}|^2 t, \\ \mathbf{v} \cdot \mathbf{w} = jk|\mathbf{r}||\mathbf{p}|^2|\mathbf{q}|t. \end{cases}$$

Since $i, j, k > 0$, and $|\mathbf{p}|, |\mathbf{q}|, |\mathbf{r}| > 0$ since none of P, Q, R are at O, we must have

$$\operatorname{sgn}(\mathbf{u} \cdot \mathbf{v}) = \operatorname{sgn}(\mathbf{u} \cdot \mathbf{w}) = \operatorname{sgn}(\mathbf{v} \cdot \mathbf{w}) = \operatorname{sgn} t,$$

where $\operatorname{sgn} : \mathbb{R} \to \{-1, 0, -1\}$ is the sign function defined as

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

But the sign of a dot product also corresponds to the angle between two non-collinear non-zero vectors, since this resembles the sign of the cosine of the angle between them:

$$\begin{aligned} \operatorname{sgn} \mathbf{a} \cdot \mathbf{b} &= \operatorname{sgn} |\mathbf{a}||\mathbf{b}| \cos\langle \mathbf{a}, \mathbf{b} \rangle \\ &= \operatorname{sgn} \cos\langle \mathbf{a}, \mathbf{b} \rangle \\ &= \begin{cases} 1, & \langle a, b \rangle \text{ is acute}, \\ 0, & \langle a, b \rangle \text{ is right-angle}, \\ -1, & \langle a, b \rangle \text{ is obtuse}. \end{cases} \end{aligned}$$

This means the angles between $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{w}$, $\mathbf{v}$ and $\mathbf{w}$ must all be acute, obtuse, or right angles. This is exactly what is desired, and finishes our proof.

2024.2 Question 5

1. We have
$$f_1(n) = n^2 + 6n + 11 = (n + 3)^2,$$
and so
$$f_1(\mathbb{Z}) = \{(n + 3)^2 + 2 \mid n \in \mathbb{Z}\}.$$
But since if $n \in \mathbb{Z}, n + 3 \in \mathbb{Z}$, and if $n + 3 \in \mathbb{Z}, n \in \mathbb{Z}$, so
$$f_1(\mathbb{Z}) = \{(n + 3)^2 + 2 \mid n \in \mathbb{Z}\} = \{n^2 + 2 \mid n \in \mathbb{Z}\}.$$
We have $F_1(\mathbb{Z}) = \{n^2 + 2 \mid n \in \mathbb{Z}\}$, and so $f_1(\mathbb{Z}) = F_1(\mathbb{Z})$, which shows f_1 and F_1 has the same range/

2. We have
$$g_1(n) = n^2 - 2n + 5 = (n - 1)^2 + 4,$$
and so
$$g_1(\mathbb{Z}) = \{(n - 1)^2 + 4 \mid n \in \mathbb{Z}\} = \{n^2 + 4 \mid n \in \mathbb{Z}\}.$$
The quadratic residues modulo 4 are 0 and 1, and so
$$f_1(\mathbb{Z}) \subseteq \{0 + 2, 1 + 2\} = \{2, 3\} \bmod 4,$$
and
$$g_1(\mathbb{Z}) \subseteq \{0 + 4, 1 + 4\} = \{0, 1\} \bmod 4.$$
Under modulo 4, $f_1(\mathbb{Z}) \cap g_1(\mathbb{Z}) \subseteq \{2, 3\} \cap \{0, 1\} = \varnothing$.
Hence, $f_1(\mathbb{Z}) \cap g_1(\mathbb{Z}) = \varnothing$ under modulo 4, and hence $f_1(\mathbb{Z}) \cap g_1(\mathbb{Z}) = \varnothing$.

3. We have
$$f_2(n) = n^2 - 2n - 6 = (n - 1)^2 - 7,$$
and so
$$f_2(\mathbb{Z}) = \{(n - 1)^2 - 7 \mid n \in \mathbb{Z}\} = \{n^2 - 7 \mid n \in \mathbb{Z}\}.$$
Similarly,
$$g_2(n) = n^2 - 4n + 2 = (n - 2)^2 - 2,$$
and so
$$g_2(\mathbb{Z}) = \{(n - 2)^2 - 2 \mid n \in \mathbb{Z}\} = \{n^2 - 2 \mid n \in \mathbb{Z}\}.$$
So for the intersection, if $t \in f_2(\mathbb{Z}) \cap g_2(\mathbb{Z})$, then there exists $n_1, n_2 \in \mathbb{Z}$,
$$t = n_1^2 - 7 = n_2^2 - 2,$$
and hence
$$n_1^2 - n_2^2 = (n_1 + n_2)(n_1 - n_2) = 5.$$
So
$$(n_1 + n_2, n_1 - n_2) = (\pm 1, \pm 5) \text{ or } (\pm 5, \pm 1),$$
and hence
$$(n_1, n_2) = (\pm 3, \mp 2) \text{ or } (\pm 3, \pm 2),$$
which gives
$$t = (\pm 3)^2 - 7 = 2.$$
Therefore,
$$f_2(\mathbb{Z}) \cap g_2(\mathbb{Z}) = \{2\},$$
and 2 is the only integer which lies in the intersection of the range of f_2 and g_2.

4. Since $p, q \in \mathbb{R}$, we must have $p+q, p-q \in \mathbb{R}$ and hence

$$(p+q)^2 = p^2 + 2pq + q^2 \geq 0,$$
$$(p-q)^2 = p^2 - 2pq + q^2 \geq 0.$$

Hence,

$$\begin{aligned}\frac{3}{4}(p+q)^2 + \frac{1}{4}(p-q)^2 &= \frac{3}{4}\left(p^2 + 2pq + q^2\right) + \frac{1}{4}\left(p^2 - 2pq + q^2\right)\\ &= p^2 + pq + q^2\\ &\geq 0,\end{aligned}$$

as desired.

We have
$$f_3(n) = n^3 - 3n^2 + 7n = (n-1)^3 + 4n + 1 = (n-1)^3 + 4(n-1) + 5,$$
and so
$$f_3(\mathbb{Z}) = \{(n-1)^3 + 4(n-1) + 5 \mid n \in \mathbb{Z}\} = \{n^3 + 4n + 5 \mid n \in \mathbb{Z}\}.$$
We have
$$g_3(\mathbb{Z}) = \{n^3 + 4n - 6 \mid n \in \mathbb{Z}\}.$$
So if $t \in f_3(\mathbb{Z}) \cap g_3(\mathbb{Z})$, then there exists $n_1, n_2 \in \mathbb{Z}$ such that

$$t = n_1^3 + 4n_1 + 5 = n_2^3 + 4n_2 - 6.$$

Hence,
$$(n_1^3 - n_2^3) + 4(n_1 - n_2) = (n_1 - n_2)(n_1^2 + n_1n_2 + n_2^2 + 4) = -11.$$
Since $n_1^2 + n_1n_2 + n_2^2 \geq 0$ by the lemma in the previous part, we have $n_1^2 + n_1n_2 + n_2^2 + 4 \geq 4$. But $n_1^2 + n_1n_2 + n_2^2 + 4 \mid -11$, and so

$$n_1^2 + n_1n_2 + n_2^2 + 4 = 11, n_1 - n_2 = -1.$$

Putting $n_2 = n_1 + 1$ into the first equation, we have

$$\begin{aligned}n_1^2 + n_1n_2 + n_2^2 + 4 &= n_1^2 + n_1(n_1+1) + (n_1+1)^2 + 4\\ &= n_1^2 + n_1^2 + n_1 + n_1^2 + 2n_1 + 1 + 4\\ &= 3n_1^2 + 3n_1 + 5\\ &= 11,\end{aligned}$$

and hence
$$3n_1^2 + 3n_1 - 6 = 3(n_1+2)(n_1-1) = 0,$$
which gives $n_1 = -2$ or $n_1 = 1$, and they correspond to $n_2 = -1$ or $n_2 = 2$.

Hence,
$$t = (-1)^3 + 4(-1) - 6 = -1 - 4 - 6 = -11,$$
or
$$t = 2^3 + 4 \cdot 2 - 6 = 8 + 8 - 6 = 10.$$

Hence,
$$f_3(\mathbb{Z}) \cap g_3(\mathbb{Z}) = \{-11, 10\},$$
and the integers that lie in the intersection of the ranges are -11 and 10.

2024.2 Question 6

1. We first look at the base case where $n = 0$, and we have

$$\text{RHS} = \frac{1}{2^{2\cdot 0}}\binom{2\cdot 0}{0} = \frac{1}{2^0}\binom{0}{0} = 1,$$

and LHS $= T_0 = 1$. So the desired statement is satisfied for the base case where $n = 0$.

Assume the original statement is true for some $n = k \geq 0$, that

$$T_n = \frac{1}{2^{2n}}\binom{2n}{n}.$$

Consider $n = k + 1$, we have

$$\begin{aligned} T_n &= T_{k+1} \\ &= \frac{2(k+1)-1}{2(k+1)}T_k \\ &= \frac{2k+1}{2(k+1)} \cdot \frac{1}{2^{2k}}\binom{2k}{k} \\ &= \frac{(2k+1)(2k+2)}{2(k+1)2(k+1)} \cdot \frac{1}{2^{2k}}\frac{(2k)!}{(}k!k!) \\ &= \frac{(2k+2)!}{(k+1)!(k+1)!} \cdot \frac{1}{2^{2k+2}} \\ &= \frac{1}{2^{2(k+1)}}\binom{2(k+1)}{k+1}, \end{aligned}$$

which is precisely the statement for $n = k + 1$.

The original statement is true for $n = 0$, and given it holds for some $n = k \geq 0$, it holds for $n = k + 1$. Hence, by the principle of mathematical induction, the statement

$$T_n = \frac{1}{2^{2n}}\binom{2n}{n}$$

holds for all integers $n \geq 0$, as desired.

2. By Newton's binomial theorem, we have

$$(1-x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x)^3 + \cdots,$$

and notice that the negative signs cancels out, and hence

$$a_n = \frac{\prod_{k=1}^{n}\frac{2k-1}{2}}{n!} = \frac{\prod_{k=1}^{n}(2k-1)}{2^n n!}.$$

Hence, we note that

$$\begin{aligned} \frac{a_r}{a_{r-1}} &= \frac{\prod_{k=1}^{r}(2k-1)/(2^r r!)}{\prod_{k=1}^{r-1}(2k-1)/(2^{r-1}(r-1)!)} \\ &= \frac{2r-1}{2r}, \end{aligned}$$

and hence

$$a_r = \frac{2r-1}{2r}a_{r-1}.$$

Note that $a_0 = 1$ as well. The sequence $\{a_n\}_0^\infty$ and $\{T_n\}_0^\infty$ have the same initial term $a_0 = T_0 = 1$, and they have the same same inductive relationship

$$a_n = \frac{2n-1}{2n}a_{n-1}, T_n = \frac{2n-1}{2n}T_{n-1}.$$

This shows they are the same sequence, hence

$$a_n = T_n$$

for all $n = 0, 1, 2, \cdots$.

3. By Newton'w binomial theorem,

$$(1-x)^{-\frac{3}{2}} = 1 + \frac{\left(-\frac{3}{2}\right)(-x)}{1!} + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-x)}{2!} + \cdots,$$

and so

$$b_n = \frac{\prod_{k=1}^{n} \frac{2k+1}{2}}{n!} = \frac{\prod_{k=1}^{n}(2k+1)}{2^n n!}.$$

Notice that

$$\begin{aligned}\frac{b_n}{a_n} &= \frac{\prod_{k=1}^{n}(2k+1)/(2^n n!)}{\prod_{k=1}^{n}(2k-1)/2^n n!} \\ &= \frac{\prod_{k=1}^{n}(2k+1)}{\prod_{k=1}^{n}(2k-1)} \\ &= \frac{\prod_{k=2}^{n+1}(2k-1)}{\prod_{k=1}^{n}(2k-1)} \\ &= \frac{2(n+1)-1}{2\cdot 1-1} \\ &= 2n+1,\end{aligned}$$

and so

$$\begin{aligned}b_n &= (2n+1)a_n \\ &= (2n+1)\cdot\frac{1}{2^{2n}}\cdot\binom{2n}{n} \\ &= \frac{2n+1}{2^{2n}}\binom{2n}{n}.\end{aligned}$$

4. By the binomial expansion, we have

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots,$$

and we have

$$(1-x)^{-\frac{1}{2}}\cdot(1-x)^{-1} = (1-x)^{-\frac{3}{2}}.$$

For a particular term in the series expansion for $(1-x)^{-\frac{3}{2}}$, say b_n, we must have

$$b_n x^n = \sum_{t=0}^{n} a_t \cdot x^t \cdot 1 \cdot x^{n-t},$$

and hence

$$b_n = \sum_{t=0}^{n} a_t,$$

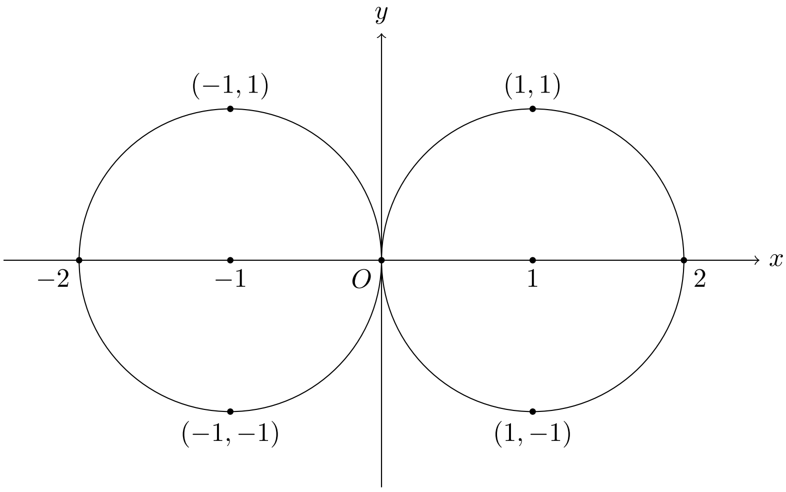
which gives

$$\frac{2n+1}{2^{2n}}\binom{2n}{n} = \sum_{r=0}^{n}\frac{1}{2^{2r}}\binom{2r}{r},$$

exactly as desired.

2024.2 Question 7

1. In this case, we have either $y^2 + (x-1)^2 = 1$ (giving a circle with radius 1 centred at $(1, 0)$), or $y^2 + (x+1)^2 = 1$ (giving a circle with radius 1 centred at $(-1, 0)$).



2. At $y = k$, we have

$$\begin{aligned}[(x-1)^2 + (k^2-1)][(x+1)^2 + (k^2-1)] &= \frac{1}{16} \\ (x-1)^2(x+1)^2 + (k^2-1)[(x-1)^2 + (x+1)^2] + (k^2-1)^2 - \frac{1}{16} &= 0 \\ (x^2-1)^2 + 2(k^2-1)(x^2+1) + (k^4 - 2k^2 + 1) - \frac{1}{16} &= 0 \\ x^4 - 2x^2 + 1 + 2(k^2-1)x^2 + 2(k^2-1) + (k^4 - 2k^2 + 1) - \frac{1}{16} &= 0 \\ x^4 + 2(k^2-2)x^2 + k^4 - \frac{1}{16} &= 0,\end{aligned}$$

as desired.

By completing the square, we have

$$\begin{aligned}(x^2 + (k^2-2))^2 + k^4 - \frac{1}{16} - (k^2-2)^2 &= 0 \\ (x^2 + (k^2-2))^2 &= \frac{65}{16} - 4k^2.\end{aligned}$$

- If $4k^2 > \frac{65}{16}$, i.e. $k^2 > \frac{65}{64}$, the right-hand side is negative, so there will be no intersections.
- If $4k^2 = \frac{65}{16}$, i.e. $k^2 = \frac{65}{64}$, we have

$$x^2 + (k^2 - 2) = 0,$$

and hence

$$x^2 = 2 - k^2 = 2 - \frac{65}{64} = \frac{63}{64},$$

giving

$$x = \pm\frac{3\sqrt{7}}{8}.$$

There will be two intersections.
- If $4k^2 < \frac{65}{16}$, i.e. $k^2 < \frac{65}{64}$, we have

$$x^2 + (k^2 - 2) = \pm\sqrt{\frac{65}{16} - 4k^2},$$

and hence

$$x^2 = 2 - k^2 \pm \sqrt{\frac{65}{16} - 4k^2}.$$

The case where

$$\begin{aligned} x^2 &= 2 - k^2 + \sqrt{\frac{65}{16} - 4k^2} \\ &> 2 - k^2 \\ &> 2 - \frac{65}{64} \\ &= \frac{63}{64} \\ &> 0 \end{aligned}$$

always gives two solutions for x.

- If $2 - k^2 - \sqrt{\frac{65}{16} - 4k^2} < 0$,

$$\begin{aligned} 2 - k^2 - \sqrt{\frac{65}{16} - 4k^2} &< 0 \\ \sqrt{\frac{65}{16} - 4k^2} &> 2 - k^2 \\ \frac{65}{16} - 4k^2 &> k^4 - 4k^2 + 4 \\ k^4 &< \frac{1}{16} \\ k^2 &< \frac{1}{4}, \end{aligned}$$

  there are no solutions for the case where the minus sign is taken.
- If $2 - k^2 - \sqrt{\frac{65}{16} - 4k^2} = 0$, $k^2 = \frac{1}{4}$, the minus sign produce precisely one solution $x = 0$, giving 3 intersections in total.
- If $2 - k^2 - \sqrt{\frac{65}{16} - 4k^2} < 0$, $k^2 > \frac{1}{4}$, the minus sign will produce two additional roots, hence giving 4 intersections in total.

To summarise, the number of intersections with the line $y = k$ for each positive value of k is

$$\text{number of intersections} = \begin{cases} 0, & k^2 > \frac{65}{64}, k > \frac{\sqrt{65}}{8}, \\ 2, & k^2 = \frac{65}{64}, k = \frac{\sqrt{65}}{8}, \\ 4, & \frac{1}{4} < k^2 < \frac{65}{64}, \frac{1}{2} < k < \frac{\sqrt{65}}{8}, \\ 3, & k^2 = \frac{1}{4}, k = \frac{1}{2}, \\ 2, & k^2 < \frac{1}{4}, 0 < k < \frac{1}{2}. \end{cases}$$

3. When the point on C_2 has the greatest possible y-coordinate, the two points have x-coordinates

$$x = \pm\frac{3\sqrt{7}}{8},$$

   and on C_1 has

$$x = \pm 1.$$

   Since $3\sqrt{7} = \sqrt{63} < \sqrt{64} = 8$, we must have $\frac{3\sqrt{7}}{8} < 1$, meaning those on C_2 are closer to the y-axis than those on C_1.

4. If both are negative, then the distance from (x, y) to $(1, 0)$ and $(-1, 0)$ are both less than 1. But this is not possible, since the distance from $(1, 0)$ to $(-1, 0)$ is 2, which means the sum of the distances from (x, y) to those points has to be at least 2.

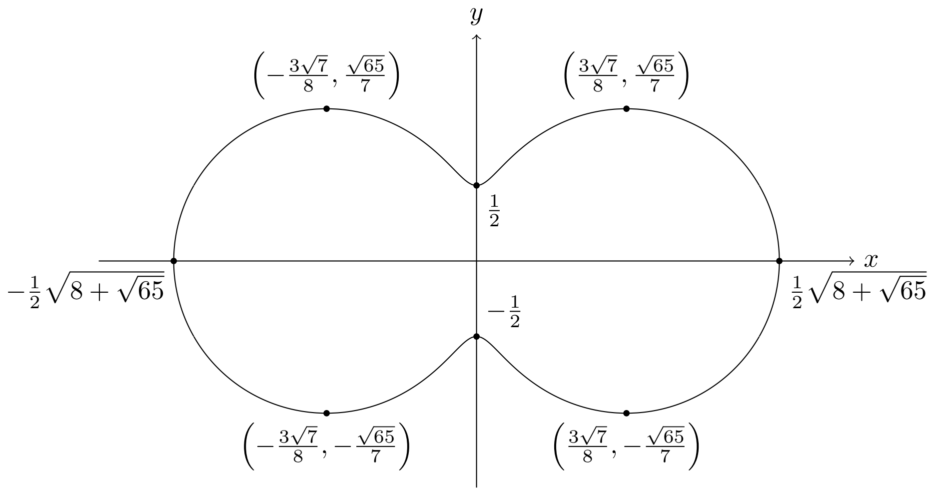
Therefore, since the product of those two terms are positive for C_2, and they cannot be both negative, they must both be positive, and hence the distance from (x, y) to $(1, 0)$ and $(-1, 0)$ are both more than 1, meaning all points on C_2 lies outside the two circles that make up C_1, which shows that C_2 lies entirely outside C_1.

5. C_2 is symmetric about both the x-axis and the y-axis.

   When $x = 0$, $y^4 = \frac{1}{16}$, and hence $y = \pm\frac{1}{2}$.

   When $y = 0$, $x^2 = 2 + \frac{\sqrt{65}}{16}$, and hence $x = \pm\sqrt{2 + \frac{\sqrt{65}}{4}} = \pm\frac{1}{2}\sqrt{8 + \sqrt{65}}$.

   Hence, the graph looks as follows.



2024.2 Question 8

1. Notice that by expanding this square,

$$\begin{aligned}(\sqrt{x_n}-\sqrt{y_n})^2 &= x_n+y_n-2\sqrt{x_ny_n}\\ &= 2a(x_n,y_n)-2g(x_n,y_n)\\ &= 2(x_{n+1}-y_{n+1}).\end{aligned}$$

Since this is a square, it must be non-negative, with the equal sign taking if and only if $\sqrt{x_n}=\sqrt{y_n}$, which holds if and only if $x_n=y_n$.

So $x_{n+1}\geq y_{n+1}$, and $x_{n+1}=y_{n+1}$ if and only if $x_n=y_n$.

Since $y_0<x_0$, we have $y_0\neq x_0$, and hence $y_1\neq x_1$. By induction, this shows that $y_n \neq_n$ for all n, and hence for all $n\geq 0$, $y_n<x_n$.

Furthermore,

$$\begin{aligned}x_n-x_{n+1} &= x_n-a(x_n,y_n)\\ &= x_n-\frac{x_n+y_n}{2}\\ &= \frac{x_n-y_n}{2}\\ &> 0,\end{aligned}$$

since $x_n>y_n$ and hence $x_n>x_{n+1}$.

Similarly,

$$\begin{aligned}y_{n+1}-y_n &= g(x_n,y_n)-y_N\\ &= \sqrt{x_ny_n}-y_N\\ &= \sqrt{y_n}(\sqrt{x_n}-\sqrt{y_n})\\ &> 0,\end{aligned}$$

since $x_n>y_n$ implies $\sqrt{x_n}>\sqrt{y_n}$, and hence $y_n<y_{n+1}$.

Hence, for all $n\in\mathbb{N}$,

$$y_n<x_n<x_{n-1}<x_{n-2}<\cdots<x_0,$$

and $y_{n+1}>y_n$.

Hence, $\{y_N\}_{n=0}^{\infty}$ is an increasing sequence, and is bounded above by x_0.

So there exists $M\in\mathbb{R}$ such that

$$\lim_{n\to\infty} y_n = M.$$

As for the inequality, the left inequality sign is equivalent to $y_{n+1}<x_{n+1}$ which was shown above.

To show the right inequality sign, this is equivalent to showing

$$\begin{aligned}\frac{1}{2}(\sqrt{x_n}-\sqrt{y_n})^2 &< \frac{1}{2}(x_n-y_n)\\ x_n+y_n-2\sqrt{x_ny_N} &< x_n-y_n\\ 2y_n &< 2\sqrt{x_ny_n}\\ \sqrt{y_n} &< \sqrt{x_n},\end{aligned}$$

which is true since $y_n<x_n$.

Hence,

$$0<x_{n+1}-y_{n+1}<\frac{1}{2}(x_n-y_n)$$

as desired.

Hence, we have

$$\begin{aligned} 0 &< x_n - y_n \\ &< \frac{1}{2}(x_{n-1} - y_{n-1}) \\ &< \frac{1}{4}(x_{n-2} - y_{n-2}) \\ &< \cdots \\ &< \frac{1}{2^n}(x_0 - y_0), \end{aligned}$$

by induction.

$x_0 - y_0 > 0$ is a positive real constant. Let $n \to \infty$, and by the squeeze theorem, the strict inequalities become weak, and

$$0 \leq \lim_{n\to\infty}(x_n - y_n) \leq \lim_{n\to\infty}\left(\frac{1}{2^n}(x_0 - y_0)\right) = 0,$$

and hence

$$\lim_{n\to\infty}(x_n - y_n) = 0.$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} x_n &= \lim_{n\to\infty}\left[(x_n - y_n) + y_n\right] \\ &= \lim_{n\to\infty}(x_n - y_n) + \lim_{n\to\infty} y_n \\ &= 0 + M \\ &= M, \end{aligned}$$

since both parts of the limit $x_n - y_n$ and y_n exist, the limit of the sum is the sum of the limits of the individual parts.

2. Using this substitution, when $x \to 0^+$, we have $t \to -\infty$, and when $x \to +\infty$, $t \to +\infty$. Also,

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2} + \frac{1}{2} \cdot \frac{pq}{x^2} = \frac{1}{2}\left(1 + \frac{pq}{x^2}\right).$$

Hence, the integral can be simplified as

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\sqrt{\left(\frac{1}{4}(p+q)^2 + t^2\right)(pq + t^2)}} \\ &= \int_0^{\infty} \frac{\frac{1}{2}\left(1 + \frac{pq}{x^2}\right)\mathrm{d}x}{\sqrt{\left(\frac{1}{4}(p+q)^2 + \frac{1}{4}\left(x - \frac{pq}{x}\right)^2\right)\left(pq + \frac{1}{4}\left(x - \frac{pq}{x}\right)^2\right)}} \\ &= \int_0^{\infty} \frac{\frac{1}{2}\left(1 + \frac{pq}{x^2}\right)\mathrm{d}x}{\frac{1}{4}\sqrt{\left(p^2 + 2pq + q^2 + x^2 - 2pq + \frac{p^2q^2}{x^2}\right)\left(4pq + x^2 - 2pq + \frac{p^2q^2}{x^2}\right)}} \\ &= 2\int_0^{\infty} \frac{\left(1 + \frac{pq}{x^2}\right)\mathrm{d}x}{\sqrt{\left(p^2 + q^2 + x^2 + \frac{p^2q^2}{x^2}\right)\left(x^2 + 2pq + \frac{p^2q^2}{x^2}\right)}} \\ &= 2\int_0^{\infty} \frac{(x^2 + pq)\,\mathrm{d}x}{\sqrt{(x^4 + (p^2+q^2)x^2 + p^2q^2)(x^4 + 2pqx^2 + p^2q^2)}} \\ &= 2\int_0^{\infty} \frac{(x^2 + pq)\,\mathrm{d}x}{\sqrt{(x^2+p^2)(x^2+q^2)(x^2+pq)^2}} \\ &= 2\int_0^{\infty} \frac{\mathrm{d}x}{\sqrt{(x^2+p^2)(x^2+q^2)}} \\ &= 2I(p,q), \end{aligned}$$

which means
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\sqrt{\left(\frac{1}{4}(p+q)^2+t^2\right)(pq+t^2)}} = 2I(p,q).$$

But also note that the left-hand side satisfies that
$$\begin{aligned}
\text{LHS} &= \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\sqrt{\left(\frac{1}{4}(p+q)^2+t^2\right)(pq+t^2)}} \\
&= 2\int_0^{\infty} \frac{\mathrm{d}t}{\sqrt{\left[\left(\frac{1}{2}(p+q)\right)^2+t^2\right]\left[(\sqrt{pq})^2+t^2\right]}} \\
&= 2\int_0^{\infty} \frac{\mathrm{d}t}{\sqrt{\left[a(p,q)^2+t^2\right]\left[g(p,q)^2+t^2\right]}} \\
&= 2I(a(p,q), g(p,q)),
\end{aligned}$$

since the integrand is an even function, and so
$$I(p,q) = I(a(p,q), g(p,q)),$$

as desired.

Since $0 < q < p$, let $y_0 = q, x_0 = p$, and hence
$$\begin{aligned}
I(p,q) &= I(x_0, y_0) \\
&= I(a(x_0,y_0), g(x_0,y_0)) \\
&= I(x_1, y_1) \\
&= \cdots \\
&= I(x_n, y_n).
\end{aligned}$$

Let $n \to \infty$, and we have
$$\begin{aligned}
I(p,q) &= I(M, M) \\
&= \int_0^{\infty} \frac{\mathrm{d}x}{M^2+x^2} \\
&= \frac{1}{M}\left[\arctan\left(\frac{x}{M}\right)\right]_0^{\infty} \\
&= \frac{\pi}{2M}.
\end{aligned}$$

2024.2 Question 11

1. Notice that
$$x^{\frac{1}{x}} = \exp\left(\frac{\ln x}{x}\right).$$

   As $x \to 0^+$, $\frac{\ln x}{x} \to -\infty$, and hence $x^{\frac{1}{x}} \to 0^+$.

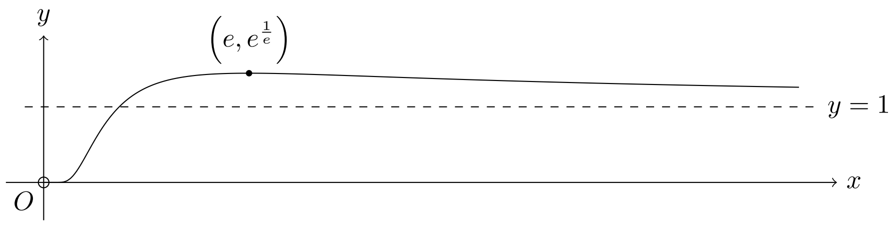
   As $x \to \infty$, $\frac{\ln x}{x} \to 0^+$, and hence $x^{\frac{1}{x}} \to 1$.

   We have
$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\exp\left(\frac{\ln x}{x}\right)\\ &= x^{\frac{1}{x}} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\frac{\ln x}{x}\\ &= x^{\frac{1}{x}} \cdot \frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2}\\ &= x^{\frac{1}{x}} \cdot \frac{1-\ln x}{x^2}.\end{aligned}$$

   This shows that $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ when $x > e$, $= 0$ when $x = e$, and > 0 when $x < e$.

   This means that the point $\left(e, e^{\frac{1}{e}}\right)$ is a maximum for the graph.

   Hence, the graph looks as follows.



   The maximum of $n^{\frac{1}{n}}$ must occur for $n \in \mathbb{N}$ when $n = 2$ or $n = 3$, since $2 < e < 3$.

   Notice that
$$\begin{aligned}2^{\frac{1}{2}} < 3^{\frac{1}{3}} &\iff 2^3 < 3^2\\ &\iff 8 < 9,\end{aligned}$$

   which is true, so the maximum of $n^{\frac{1}{n}}$ occurs when $n = 3$.

2. Let X_i be the number of tests for each group, and let X be the total number of tests, we have
$$X = \sum_{i=1}^{r} X_i.$$

   For each X_i, we have if the enzyme is not present in any of the persons, then there is only one test needed. Otherwise, if the enzyme is present in any of the persons, then an additional k tests are needed. Hence,
$$\mathrm{E}(X_i) = (1-p)^k + (1-(1-p)^k)(1+k) = 1 + (1-p^k)k,$$

and the expected total number of tests is given as

$$\begin{aligned}
\mathrm{E}(X) &= \mathrm{E}\left(\sum_{i=1}^{r} X_i\right) \\
&= \sum_{i=1}^{r} \mathrm{E}\left(X_i\right) \\
&= \sum_{i=1}^{r} \left[1 + (1 - (1-p)^k)k\right] \\
&= r\left[1 + (1 - (1-p)^k)k\right] \\
&= \frac{N}{k}\left[1 + (1 - (1-p)^k)k\right] \\
&= N\left(\frac{1}{k} + 1 - (1-p)^k\right).
\end{aligned}$$

3. The expected number of tests is at most N is the equation

$$\begin{aligned}
N\left(\frac{1}{k} + 1 - (1-p)^k\right) &\leq N \\
\frac{1}{k} + 1 - (1-p)^k &\leq 1 \\
\frac{1}{k} &\leq (1-p)^k \\
\left(\frac{1}{k}\right)^{\frac{1}{k}} &\leq 1 - p \\
\frac{1}{1-p} &\leq k^{\frac{1}{k}}.
\end{aligned}$$

The maximum of $k^{\frac{1}{k}}$ arises where $k = 3$, and this is valid since $k = 3 \mid N$. Hence,

$$\begin{aligned}
\frac{1}{1-p} &\leq 3^{\frac{1}{3}} \\
p &\leq 1 - 3^{-\frac{1}{3}},
\end{aligned}$$

and hence such largest value of p is

$$p = 1 - 3^{-\frac{1}{3}}.$$

Notice that

$$\begin{aligned}
1 - 3^{-\frac{1}{3}} > \frac{1}{4} &\iff \frac{3}{4} > 3^{-\frac{1}{3}} \\
&\iff \left(\frac{3}{4}\right)^3 > 3^{-1} \\
&\iff \frac{27}{64} > \frac{1}{3} \\
&\iff 81 > 64,
\end{aligned}$$

which is true, and so this value of p is greater than $\frac{1}{4}$.

4. We would like to show that if $pk \ll 1$, then $1 - (1-p)^k \approx pk$.
Notice that

$$\begin{aligned}
1 - (1-p)^k &= 1 - \sum_{i=0}^{k} \binom{k}{i} (-p)^k \\
&= 1 - (1 - kp + \cdots) \\
&\approx kp,
\end{aligned}$$

and hence

$$\mathrm{E}(X) = N\left(\frac{1}{k} + 1 - (1-p)^k\right) \approx N\left(\frac{1}{k} + pk\right).$$

If $p = 0.01$, $k = 10$, we have

$$\mathrm{E}(X) \approx N\left(\frac{1}{10} + 0.01 \cdot 10\right) = N \cdot \frac{2}{10} = \frac{1}{5}N,$$

which is 20% of N.

2024.2 Question 12

1. Let X_i be the number that the ith player receives, and let Ada be the first player. We have

$$\begin{aligned}
\mathrm{P}(X_1 = a, X_2 > X_1, X_3 > X_1, \cdots, X_k > X_1) &= \mathrm{P}(X_1 = a, X_2 > a, X_3 > a, \cdots, X_k > a) \\
&= \mathrm{P}(X_1 = a)\,\mathrm{P}(X_2 > a)\,\mathrm{P}(X_3 > a) \cdots \mathrm{P}(X_k > a) \\
&= \frac{1}{n} \cdot \frac{n-a}{n} \cdot \frac{n-a}{n} \cdots \frac{n-a}{n} \\
&= \frac{(n-a)^{k-1}}{n^k}.
\end{aligned}$$

Hence, the probability of Ada winning this is

$$\begin{aligned}
\mathrm{P}(X_2 > X_1, X_3 > X_1, \cdots, X_k > X_1) &= \sum_{a=1}^{n-1} \mathrm{P}(X_1 = a, X_2 > X_1, X_3 > X_1, \cdots, X_k > X_1) \\
&= \sum_{a=1}^{n-1} \frac{(n-a)^{k-1}}{n^k} \\
&= \frac{1}{n^k} \sum_{a=1}^{n-1} a^{k-1},
\end{aligned}$$

and the probability of there being a winner is the sum of the probabilities of each player winning, which are all equal to the probability of Ada winning by symmetry, and hence is equal to

$$k \cdot \frac{1}{n^k} \sum_{a=1}^{n-1} a^{k-1} = \frac{k}{n^k} \sum_{a=1}^{n-1} a^{k-1}.$$

If $k = 4$, then this probability is given by

$$\begin{aligned}
\mathrm{P} &= \frac{4}{n^4} \sum_{a=1}^{n-1} a^3 \\
&= \frac{4}{n^4} \cdot \frac{(n-1)^2 n^2}{4} \\
&= \frac{(n-1)^2}{n^2},
\end{aligned}$$

precisely as desired.

2. Similarly, let X_i be the number that the ith player receives, and let Ada be the first player, and Bob be the second player. We have

$$\begin{aligned}
&\mathrm{P}(X_1 = a, X_2 = a + d + 1, X_1 < X_3 < X_2, \cdots, X_1 < X_k < X_2) \\
&= \mathrm{P}(X_1 = a, X_2 = a + d + 1, a < X_3 < a + d + 1, \cdots, a < X_k < a + d + 1) \\
&= \mathrm{P}(X_1 = a)\,\mathrm{P}(X_2 = a + d + 1)\,\mathrm{P}(a < X_3 < a + d + 1) \cdots \mathrm{P}(a < X_k < a + d + 1) \\
&= \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{d}{n} \cdots \frac{d}{n} \\
&= \frac{d^{k-2}}{n^k}.
\end{aligned}$$

Hence, the probability that both Ada and Bob winning this is

$$\begin{aligned}
&\mathrm{P}(X_1 < X_3 < X_2, \cdots, X_1 < X_k < X_2) \\
&= \sum_{d=1}^{n-2} \sum_{a=1}^{n-d-1} \mathrm{P}(X_1 = a, X_2 = a + d + 1, X_1 < X_3 < X_2, \cdots, X_1 < X_k < X_2) \\
&= \sum_{d=1}^{n-2} \sum_{a=1}^{n-d-1} \frac{d^{k-2}}{n^k} \\
&= \sum_{d=1}^{n-2} \frac{(n-d-1)d^{k-2}}{n^k} \\
&= \frac{1}{n^k} \sum_{d=1}^{n-2} (n-d-1)d^{k-2} \\
&= \frac{1}{n^k} \left[(n-1) \sum_{d=1}^{n-2} d^{k-2} - \sum_{d=1}^{n-2} d^{k-1} \right].
\end{aligned}$$

Hence, the probability that there are two winners in this game is the sum of the probabilities of each ordered pair of players winning (since there is one winning by having a bigger number, and one winning by having a smaller number), and hence is equal to

$$2 \cdot \binom{k}{2} \cdot \frac{1}{n^k} \left[(n-1) \sum_{d=1}^{n-2} d^{k-2} - \sum_{d=1}^{n-2} d^{k-1} \right].$$

When $k = 4$, the probability is

$$\begin{aligned}
\mathrm{P} &= 2 \cdot \binom{4}{2} \cdot \frac{1}{n^4} \left[(n-1) \sum_{d=1}^{n-2} d^2 - \sum_{d=1}^{n-2} d^3 \right] \\
&= 2\dot{6} \cdot \frac{1}{n^4} \left[\frac{(n-1)(n-2)(n-1)(2n-3)}{6} - \frac{(n-2)^2(n-1)^2}{4} \right] \\
&= 12 \cdot \frac{1}{n^4} \cdot (n-1)^2(n-2) \left[\frac{2(2n-3) - 3(n-2)}{12} \right] \\
&= \frac{(n-1)^2(n-2)}{n^4} \cdot n \\
&= \frac{(n-2)(n-1)^2}{n^3}.
\end{aligned}$$

The probability of there being a winner due to having the biggest number (denote this event as B), is the same as there being a winner due to having the lowest number (denote this event as L), which are both equal to the answer to the first part of the question:

$$\mathrm{P}(B) = \mathrm{P}(L) = \frac{(n-1)^2}{n^2}.$$

The event of having two winners is B, L and the event of having precisely one winner is $B, \bar{L}$ or $L, \bar{B}$. By the inclusion-exclusion principle, the probability of having precisely one winner is given by

$$\begin{aligned}
\mathrm{P} &= \mathrm{P}(B) + \mathrm{P}(L) - 2\,\mathrm{P}(B, L) \\
&= 2 \cdot \frac{(n-1)^2}{n^2} - 2 \cdot \frac{(n-2)(n-1)^2}{n^3} \\
&= \frac{2(n-1)^2}{n^3} \cdot [n - (n-2)] \\
&= \frac{4(n-1)^2}{n^3}.
\end{aligned}$$

This probability is smaller than $\mathrm{P}(B, L)$, if and only if

$$\begin{aligned}\frac{4(n-1)^2}{n^3} &< \frac{(n-2)(n-1)^2}{n^3} \\ 4 &< n-2 \\ n &> 6,\end{aligned}$$

and hence the minimum value of n for this is 7.

2024 Paper 3

2024.3 Question 1

1. For the first identity, notice that

$$\begin{aligned}\frac{1}{r+1}-\frac{1}{r}+\frac{1}{r^2}&=\frac{r^2-r(r+1)+(r+1)}{r^2(r+1)}\\&=\frac{r^2-r^2-r+r+1}{r^2(r+1)}\\&=\frac{1}{r^2(r+1)},\end{aligned}$$

and hence using this,

$$\begin{aligned}\sum_{r=1}^{N}\frac{1}{r^2(r+1)}&=\sum_{r=1}^{N}\left(\frac{1}{r+1}-\frac{1}{r}+\frac{1}{r^2}\right)\\&=\sum_{r=1}^{N}\frac{1}{r^2}+\sum_{r=1}^{N}\frac{1}{r+1}-\sum_{r=1}^{N}\frac{1}{r}\\&=\sum_{r=1}^{N}\frac{1}{r^2}+\sum_{r=2}^{N+1}\frac{1}{r}-\sum_{r=1}^{N}\frac{1}{r}\\&=\sum_{r=1}^{N}\frac{1}{r^2}-\frac{1}{1}+\frac{1}{N+1}\\&=\sum_{r=1}^{N}\frac{1}{r^2}-1+\frac{1}{N+1}.\end{aligned}$$

Let $N\to\infty$, and we have $\frac{1}{N+1}\to 0$, and hence

$$\sum_{r=1}^{\infty}\frac{1}{r^2(r+1)}=\sum_{r=1}^{\infty}\frac{1}{r^2}-1=\frac{\pi^2}{6}-1.$$

2. By partial fractions, let

$$\frac{1}{r^2(r+1)(r+2)}=\frac{Ar+B}{r^2}+\frac{C}{r+1}+\frac{D}{r+2}$$

for real constants A, B, C and D.

Hence,

$$(Ar+B)(r+1)(r+2)+Cr^2(r+2)+Dr^2(r+1)=1.$$

Let $r=-2$, we have $D\cdot(-2)^2\cdot(-1)=-4D=1$, and hence $D=-\frac{1}{4}$.
Let $r=-1$, we have $C\cdot(-1)^2\cdot 1=C=1$, and hence $C=1$.
Let $r=0$, we have $B\cdot 1\cdot 2=1$, and hence $B=\frac{1}{2}$.
Considering the coefficient of r^3, we have $A+C+D=0$, and hence $A=-\frac{3}{4}$.
Hence,

$$\frac{1}{r^2(r+1)(r+2)}=-\frac{3}{4}\cdot\frac{1}{r}+\frac{1}{2}\cdot\frac{1}{r^2}+\frac{1}{r+1}-\frac{1}{4}\cdot\frac{1}{r+2}.$$

Therefore,

$$\begin{aligned}\sum_{r=1}^{N}\frac{1}{r^2(r+1)(r+2)} &= -\frac{3}{4}\sum_{r=1}^{N}\frac{1}{r}+\frac{1}{2}\sum_{r=1}^{N}\frac{1}{r^2}+\sum_{r=1}^{N}\frac{1}{r+1}-\frac{1}{4}\sum_{r=1}^{N}\frac{1}{r+2}\\ &= \frac{1}{2}S_N-\frac{3}{4}\cdot\sum_{r=1}^{N}\frac{1}{r}+\sum_{r=2}^{N+1}\frac{1}{r}-\frac{1}{4}\sum_{r=3}^{N+2}\frac{1}{r}\\ &= \frac{1}{2}S_N-\frac{3}{4}\sum_{r=3}^{N}\frac{1}{r}+\sum_{r=3}^{N}\frac{1}{r}-\frac{1}{4}\sum_{r=3}^{N}\frac{1}{r}\\ &= \frac{1}{2}S_N-\frac{3}{4}\left(\frac{1}{1}+\frac{1}{2}\right)+\left(\frac{1}{2}+\frac{1}{N+1}\right)-\frac{1}{4}\left(\frac{1}{N+1}+\frac{1}{N+2}\right)\\ &= \frac{1}{2}S_N-\frac{9}{8}+\frac{4}{8}+\frac{3}{4}\cdot\frac{1}{N+1}-\frac{1}{4}\cdot\frac{1}{N+2}\\ &= \frac{1}{2}S_N-\frac{5}{8}+\frac{3}{4}\cdot\frac{1}{N+1}-\frac{1}{4}\cdot\frac{1}{N+2}.\end{aligned}$$

Let $N\to\infty$, we have $\frac{1}{N+1},\frac{1}{N+2}\to 0$, and hence

$$\sum_{r=1}^{\infty}\frac{1}{r^2(r+1)(r+2)}=\frac{1}{2}\lim_{N\to\infty}S_N-\frac{5}{8}=\frac{\pi^2}{12}-\frac{5}{8}.$$

3. Similarly, let

$$\frac{1}{r^2(r+1)^2}=\frac{A}{r^2}+\frac{B}{r}+\frac{C}{(r+1)^2}+\frac{D}{r+1}$$

for some real constants A,B,C and D.

Hence,

$$1=A(r+1)^2+Br(r+1)^2+Cr^2+Dr^2(r+1).$$

Let $r=0$, and we have $A=1$. Let $r=-1$, and we have $C=1$. Considering the coefficient of r^3 we have $B+D=0$, and for r, $2A+B=0$.

Hence, $B=-2, D=2$, and

$$\frac{1}{r^2(r+1)^2}=\frac{1}{r^2}-\frac{2}{r}+\frac{1}{(r+1)^2}+\frac{2}{r+1}.$$

Therefore, for natural numbers N, we have

$$\begin{aligned}\sum_{r=1}^{N}\frac{1}{r^2(r+1)^2} &= \sum_{r=1}^{N}\frac{1}{r^2}+\sum_{r=1}^{N}\frac{1}{(r+1)^2}+2\sum_{r=1}^{N}\frac{1}{r+1}-2\sum_{r=1}^{N}\frac{1}{r}\\ &= S_N+\sum_{r=1}^{N+1}\frac{1}{r^2}+2\sum_{r=2}^{N+1}\frac{1}{r}-2\sum_{r=1}^{N}\frac{1}{r}\\ &= S_N+S_{N+1}-\frac{1}{1^2}+2\cdot\frac{1}{N+1}-2\cdot 1\\ &= S_N+s_{N+1}+2\cdot\frac{1}{N+1}-3.\end{aligned}$$

Let $N \to \infty$. $S_N, S_{N+1} \to \frac{\pi^2}{6}$, and $\frac{1}{N+1} \to 0$. Hence,

$$\begin{aligned}\sum_{r=1}^{\infty} \frac{1}{r^2(r+1)^2} &= 2 \cdot \frac{\pi^2}{6} - 3 \\ &= \frac{\pi^2}{3} - 3 \\ &= 2 \cdot \left(\frac{\pi^2}{6} - 1\right) - 1 \\ &= 2 \sum_{r=1}^{\infty} \frac{1}{r^2(r+1)} - 1 \\ &= \sum_{r=1}^{\infty} \frac{2}{r^2(r+1)} - 1,\end{aligned}$$

as desired.

2024.3 Question 2

1. (a) We have

$$\sqrt{4x^2 - 8x + 64} \leq |x + 8| \iff 0 \leq 4x^2 - 8x + 64 \leq (x + 8)^2.$$

The left inequality can be simplified as follows:

$$\begin{aligned} 4x^2 - 8x + 64 &\geq 0 \\ x^2 - 2x + 16 &\geq 0 \\ (X - 1)^2 + 15 &\geq 0, \end{aligned}$$

which is always true.
The right inequality can be simplified as follows:

$$\begin{aligned} 4x^2 - 8x + 64 &\leq (x + 8)^2 \\ 4x^2 - 8x + 64 &\leq x^2 + 16x + 64 \\ 3x^2 - 24x &\leq 0 \\ x(x - 8) &\leq 0, \end{aligned}$$

which solves to $0 \leq x \leq 8$.
Hence, the solution to the original inequality is $x \in [0, 8]$.

(b) WE have

$$\sqrt{4x^2 - 8x + 64} \leq |3x - 8| \iff 0 \leq 4x^2 - 8x + 64 \leq (3x - 8)^2.$$

The left inequality is always true from the previous part.
The right inequality can be simplified as follows:

$$\begin{aligned} 4x^2 - 8x + 64 &\leq (3x - 8)^2 \\ 4x^2 - 8x + 64 &\leq 9x^2 - 48x + 64 \\ 5x^2 - 40x &\geq 0 \\ x(x - 8) &\geq 0, \end{aligned}$$

which solves to $x \leq 0$ or $x \geq 8$.
Hence, the solution to the original inequality is $x \in (-\infty, 0] \cup [8, \infty)$.

2. (a) We have

$$\begin{aligned} \left(\sqrt{4x^2 - 8x + 64} + 2(x - 1)\right) f(x) &= \left(\sqrt{4x^2 - 8x + 64}\right)^2 - [2(x - 1)]^2 \\ &= \left(4x^2 - 8x + 64\right) - 4\left(x^2 - 2x + 1\right) \\ &= \left(4x^2 - 8x + 64\right) - \left(4x^2 - 8x + 4\right) \\ &= 60. \end{aligned}$$

Hence,

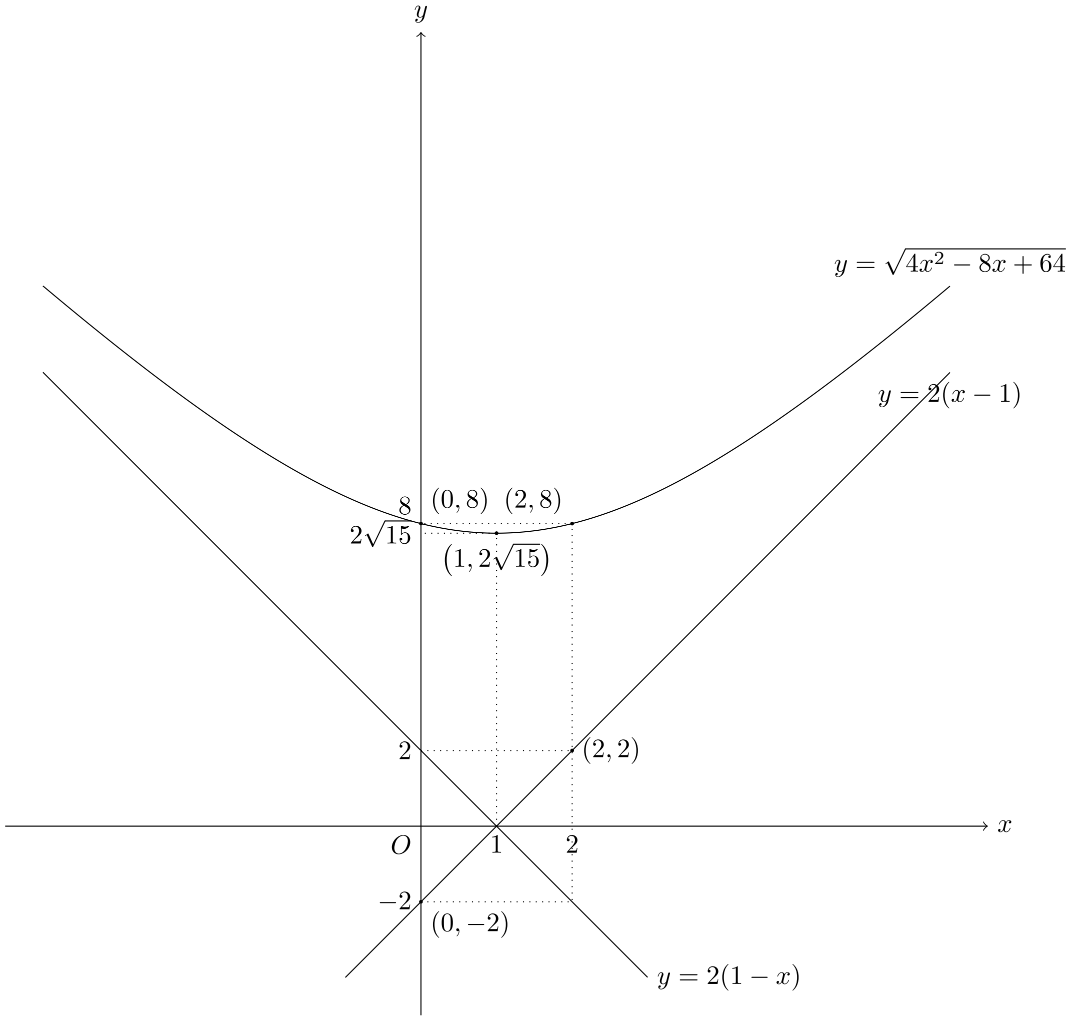
$$f(x) = \frac{60}{\sqrt{4x^2 - 8x + 64} + 2(x - 1)}.$$

As $x \to \infty$, $\sqrt{4x^2 - 8x + 64} \to \infty$, $2(x - 1) \to \infty$.
Hence, $f(x) \to 0$ as $x \to \infty$.

(b) Let $f_1(x) = \sqrt{4x^2 - 8x + 64}$, $f_2(x) = 2(x - 1)$.
$f_1(0) = \sqrt{64} = 8$, and $f_2(0) = 2 \cdot (-1) = -2$.
We have $f(x) = f_1(x) - f_2(x) > 0$ from the previous part, and that $f_1(x)$ is defined for all x and is always positive.
Furthermore,

$$f_1(x) = 2\sqrt{x^2 - 2x + 16} = 2\sqrt{(x - 1)^2 + 15},$$

and hence f_1 decreases on $(-\infty, 1)$ and increases on $(1, \infty)$, taking a minimum of $f_1(1) = 2\sqrt{15}$.
In terms of symmetry, we have $f_1(1 - x) = f_1(1 + x)$ and $f_2(1 - x) = -f_2(1 + x)$. f_2 is an asymptote to f_1 as $x \to \infty$, and $-f_2$ is an asymptote to f_1 as $x \to -\infty$.
Hence, the sketch looks as follows.



3. Let $x = 3$, and we must have $\sqrt{4 \cdot 9 - 5 \cdot 3 + 4} = |3m + c|$, and hence $5 = |3m + c|$.

   This is only achievable for $m = \pm 2$ due to the diagram − the solution set can only be 'one-sided' if on the other side the absolute value is eventually 'parallel' to the curve.

   We let $m = 2$, and hence $5 = |6 + c|$, which gives $c = -1$ or $c = -11$.

   We would like to show that the desired value is $c = -1$, and that $c = -11$ does not work.

$$\sqrt{4x^2 - 5x + 4} \leq |2x - 1| \iff 0 \leq 4x^2 - 5x + 4 \leq (2x - 1)^2.$$

   The left inequality can be simplified as

$$0 \leq 4x^2 - 5x + 4 = \left(2x - \frac{5}{4}\right)^2 + \frac{39}{16},$$

   and hence is always true.

   The right inequality can be simplified as

$$\begin{aligned} 4x^2 - 5x + 4 &\leq (2x - 1)^2 \\ 4x^2 - 5x + 4 &\leq 4x^2 - 4x + 1 \\ x &\geq 3, \end{aligned}$$

   and hence the solution set to the whole inequality is $x \geq 3$ as desired.

   On the other hand, for the case of $c = -11$, we have

$$\sqrt{4x^2 - 5x + 4} \leq |2x - 11| \iff 0 \leq 4x^2 - 5x + 4 \leq (2x - 11)^2,$$

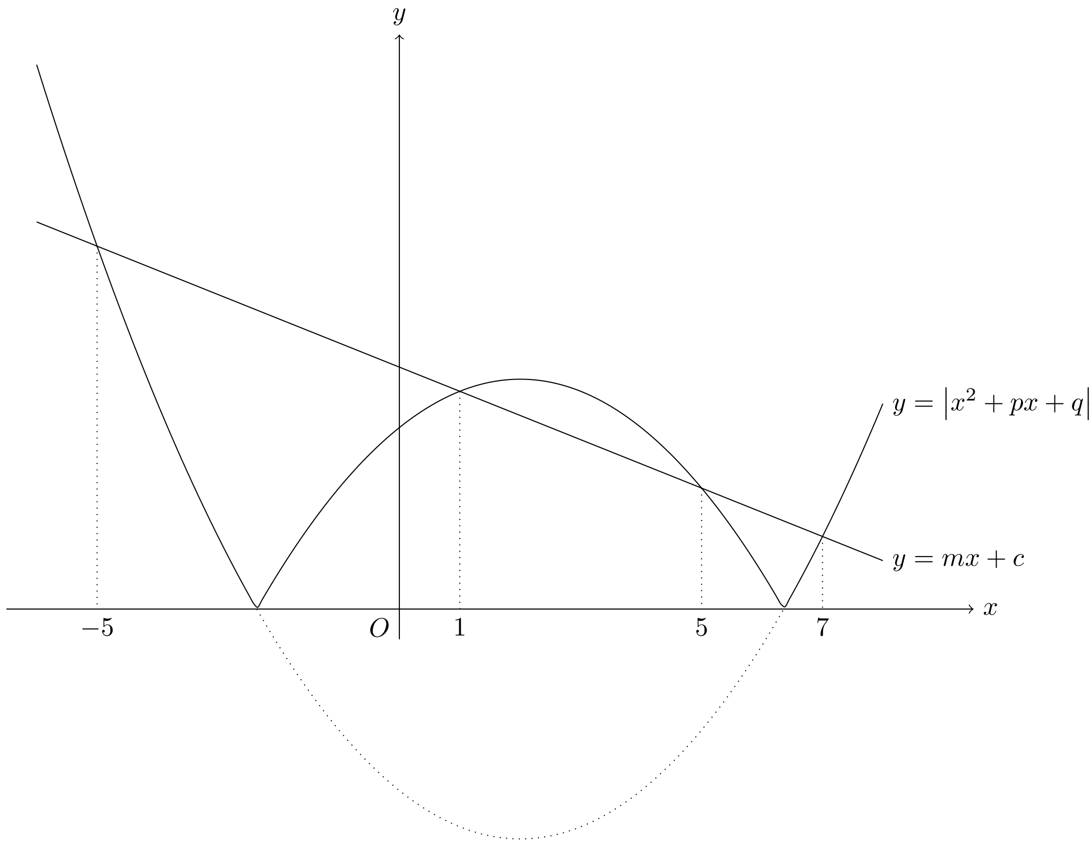
and the left inequality is always true by previously. However, the right inequality simplifies as

$$\begin{aligned} 4x^2 - 5x + 4 &\leq (2x - 11)^2 \\ 4x^2 - 5x + 4 &\leq 4x^2 - 44x + 121 \\ 39x &\leq 117 \\ x &\leq 3, \end{aligned}$$

and the inequality is in the wrong direction.

Hence, a possible value of m is 2, and the corresponding value of c is -1.

4. The diagram as follows shows the only possibility of the configuration.



Hence, we must have $x^2 + px + q = mx + c$ for $x = -5$ and $x = 7$, and $x^2 + px + q = -mx - c$ for $x = 1$ and $x = 5$.

$$\begin{cases} 25 - 5p + q = -5m + c, \\ 49 + 7p + q = 7m + c, \\ \quad 1 + p + q = -(m + c), \\ 25 + 5p + q = -(5m + c). \end{cases}$$

Subtracting the first equation from the final equation gives $10p = -2c$, and hence $c = -5p$.

Subtracting the first equation from the second equation gives us $24 + 12p = 12m$, and hence $m = 2 + p$.

Putting these into the third equation gives

$$\begin{aligned} q &= -m - c -; -1 \\ &= -(2 + p) - (-5p) - p - 1 \\ &= 3p - 3. \end{aligned}$$

Putting all these into the final equation gives

$$\begin{aligned}25 + 5p + (3p - 3) &= -\left[5(2 + p) + (-5p)\right]\\25 + 8p - 3 &= -(10 + 5p - 5p)\\22 + 8p &= -10\\8p &= -32\\p &= -4,\end{aligned}$$

and so $q = -15, m = -2, c = 20$. Hence,

$$(p, q, m, c) = (-4, -15, -2, 20).$$

2024.3 Question 3

1. (a) Notice that by partial fractions, we have

$$\frac{x+c}{x(x+1)} = \frac{1-c}{x+1} + \frac{c}{x}.$$

Hence, by differentiating, we have

$$\begin{aligned}
g'(x) &= \frac{1}{1+\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) + \frac{1-c}{(x+1)^2} + \frac{c}{x^2} \\
&= -\frac{1}{x^2+x} + \frac{1-c}{(x+1)^2} + \frac{c}{x^2} \\
&= \frac{-x(x+1) + (1-c)x^2 + c(x+1)^2}{(x+1)^2 x^2} \\
&= \frac{cx^2 + 2cx + c + x^2 - cx^2 - x^2 - x}{(x+1)^2 x^2} \\
&= \frac{(2c-1)x + c}{(x+1)^2 x^2}.
\end{aligned}$$

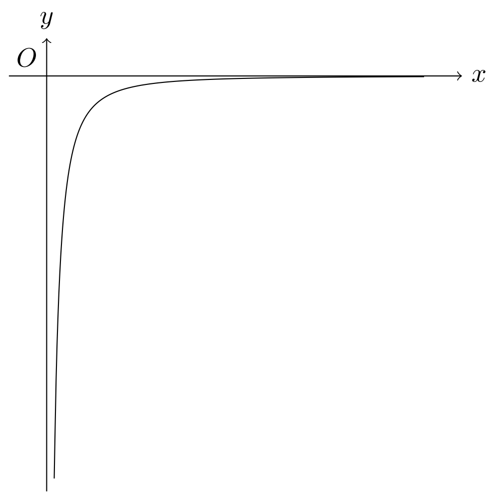
Given that $c \geq \frac{1}{2}$, and $x > 0$, we have $2c - 1 \geq 0$, and $(2c-1)x \geq 0$.
Hence, the numerator satisfies $(2c-1)x + c \geq c \geq \frac{1}{2} > 0$, and the denominator is always positive since is a product of squares, and both squares are non-zero since $x > 0$.
We can now conclude that $g'(x) > 0$ given $c \geq \frac{1}{2}$ for $x > 0$, as desired.

(b) If $0 \leq c < \frac{1}{2}$, $g'(x) < 0$ if and only if

$$\begin{aligned}
(2c-1)x + c &< 0 \\
(1-2c)x - c &> 0 \\
(1-2c)x &> c \\
x &> \frac{c}{1-2c},
\end{aligned}$$

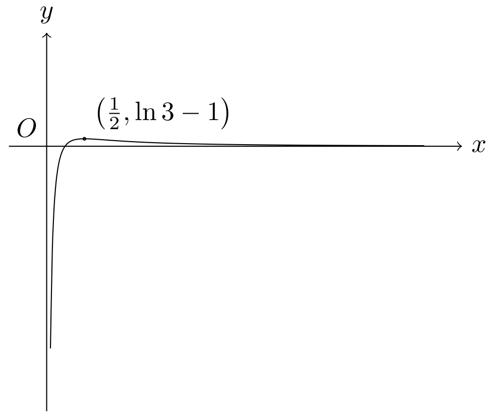
and the values of x are $x > \frac{c}{1-2c}$.

2. (a) If $c = \frac{3}{4} \geq \frac{1}{2}$, we can see that g is always increasing.
As $x \to \infty$, $\frac{x+c}{x(x+1)} \to 0$, $\ln\left(1 + \frac{1}{x}\right) \to \ln 1 = 0$. Hence, $g(x) \to 0$.
Since g is increasing it must stay entirely below the x-axis.
The sketch is as follows.



(b) If $c = \frac{1}{4} \in \left[0, \frac{1}{2}\right)$, it must be the case that $g'(x) > 0$ for $0 < x < \frac{c}{1-2c} = \frac{1}{2}$, and $g'(x) < 0$ for $x > \frac{1}{2}$.

Hence, $x = \frac{1}{2}$ is a maximum on the graph, and the corresponding y-coordinate is $g\left(\frac{1}{2}\right) = \ln 3 - 1$.
Similarly, as $x \to \infty$, $g(x) \to 0$.
The sketch is as follows.



3. We have

$$\begin{aligned}
f(x) &= \left(1 + \frac{1}{x}\right)^{x+c} \\
\ln f(x) &= (x+c) \ln \left(1 + \frac{1}{x}\right) \\
\frac{f'(x)}{f(x)} &= \ln (1+x) - (x+c) \frac{1}{x(x+1)} \\
\frac{f'(x)}{f(x)} &= g(x) \\
f'(x) &= f(x) g(x).
\end{aligned}$$

$f(x)$ is positive for $x > 0$, and hence $f'(x)$ takes the same sign as $g(x)$.

(a) If $c \geq \frac{1}{2}$, g is increasing and has a limit of 0 at infinity. Hence, $g(x)$ is negative for all $x > 0$, which means $f'(x)$ is negative for all $x > 0$, and hence f is decreasing.

(b) If $0 < c < \frac{1}{2}$, g is negative first, then increases to a positive value, and remains positive and approaches 0 decreasing from above. Hence, f' is first positive and then negative, so f must have a turning point.

(c) If $c = 0$,

$$g'(x) = \frac{-x}{(x+1)^2 x^2} = -\frac{1}{(x+1)^2 x}$$

is always negative, and $\lim_{x \to 0^+} g'(x) = -\infty$, $\lim_{x \to \infty} g'(x) = 0$.
We have

$$g(x) = \ln \left(1 + \frac{1}{x}\right) - \frac{1}{x+1}.$$

As $x \to 0^+$, $\frac{1}{x} \to \infty$, so $\ln\left(1 + \frac{1}{x}\right) \to \infty$, and $-\frac{1}{x+1} \to -\frac{1}{1} = -1$. Hence, $g(x) \to \infty$.
As $x \to \infty$, $g(x) \to 0$.
Since g is decreasing, it must be the case that g is always positive.
This means that f' is always positive as well, and hence f is increasing.

2024.3 Question 4

1. The angle between a line with gradient m and the positive x-axis is $\arctan m$. Hence, we must have

$$\begin{aligned}\arctan m_1 - \arctan m_2 &= \pm\frac{\pi}{4}\\ \tan\left(\arctan m_1 - \arctan m_2\right) &= \tan\left(\pm\frac{\pi}{4}\right)\\ \frac{m_1 - m_2}{1 + m_1 m_2} &= \pm 1,\end{aligned}$$

as desired.

2. We have $y = \frac{x^2}{4a}$, and hence $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{2a}$. Hence, the tangent to the point $\left(p, \frac{p^2}{4a}\right)$ is given by

$$\begin{aligned}y - \frac{p^2}{4a} &= \frac{p}{2a}\left(x - p\right)\\ 4ay - p^2 &= 2p(x - p)\\ 4ay &= 2px - p^2,\end{aligned}$$

with gradient $\frac{2p}{4a} = \frac{p}{2a}$, and the tangent to the point $\left(q, \frac{q^2}{4a}\right)$ is given by $4ay = 2qx + q^2$, with gradient $\frac{q}{2a}$.

Hence, when they intersect, it must be the case that

$$\begin{aligned}2px - p^2 &= 2qx - q^2\\ 2(p-q)x &= p^2 - q^2\\ 2(p-q)x &= (p+q)(p-q)\\ x &= \frac{p+q}{2}\end{aligned}$$

since $p \neq q$.

The y-coordinate is given by

$$\begin{aligned}y &= \frac{2px - p^2}{4a}\\ &= \frac{p^2 + pq - p^2}{4a}\\ &= \frac{pq}{4a}.\end{aligned}$$

If the two curves meet at $\frac{\pi}{4}$, the gradients must satisfy that

$$\begin{aligned}\frac{\frac{p}{2a} - \frac{q}{2a}}{1 + \frac{p}{2a}\cdot\frac{q}{2a}} &= \pm 1\\ \frac{2a(p-q)}{4a^2 + pq} &= \pm 1\\ 2a(p-q) &= \pm\left(4a^2 + pq\right)\\ 4a^2(p-q)^2 &= (4a^2 + pq)^2\\ 4a^2p^2 - 8a^2pq + 4a^2q^2 &= 16a^4 + 8a^2pq + p^2q^2\\ p^2q^2 + 16a^2pq + 16a^4 - 4a^2p^2 - 4a^2q^2 &= 0.\end{aligned}$$

For the intersection of the two tangents, we consider $(y+3a)^2 - (8a^2 + x^2)$.

$$\begin{aligned}(y+3a)^2 - (8a^2 + x^2) &= y^2 + 6ay + 9a^2 - 8a^2 - x^2\\ &= y^2 + 6ay - x^2 + a^2\\ &= \frac{p^2q^2}{16a^2} + 6a\cdot\frac{pq}{4a} - \left(\frac{p+q}{2}\right)^2 + a^2\\ &= \frac{p^2q^2}{16a^2} + \frac{3pq}{2} - \frac{(p+q)^2}{4} + a^2.\end{aligned}$$

We have the following being equivalent:

$$
\begin{aligned}
(y+3a)^2 &= 8a^2 + x^2 \\
\frac{p^2q^2}{16a^2} + \frac{3pq}{2} - \frac{(p+q)^2}{4} + a^2 &= 0 \\
p^2q^2 + 3pq \cdot 8a^2 - (p+q)^2 \cdot 4a^2 + a^2 \cdot 16a^2 &= 0 \\
p^2q^2 + 24pqa^2 - 4a^2p^2 - 4a^2q^2 - 8pqa^2 + 16a^4 &= 0 \\
p^2q^2 + 16a^2pq + 16a^4 - 4a^2p^2 - 4a^2q^2 &= 0,
\end{aligned}
$$

which was true due to the tangents intersecting at $\frac{\pi}{4}$.

Hence, we must have the intersection of two tangents lie on $(y+3a)^2 = 8a^2 + x^2$, which finishes our proof.

3. Let θ be this acute angle, and from the previous part, we can see that

$$
\begin{aligned}
4a^2(p-q)^2 &= \tan^2\theta(4a^2+pq)^2 \\
4a^2p^2 - 8a^2pq + 4a^2q^2 &= \tan^2\theta 16a^4 + \tan^2\theta 8a^2pq + \tan^2\theta p^2q^2 \\
\tan^2\theta p^2q^2 + 8(\tan^2\theta+1)a^2pq + \tan^2\theta 16a^4 &= 4a^2p^2 + 4a^2q^2
\end{aligned}
$$

Given $(y+7a)^2 = 48a^2 + 3x^2$ for the intersection of the two tangents, we have

$$
\begin{aligned}
(y+7a)^2 - \left(48a^2+3x^2\right) &= 0 \\
\left(\frac{pq}{4a} + 7a\right)^2 - \left(48a^2 + 3\left(\frac{p+q}{2}\right)^2\right) &= 0 \\
\frac{p^2q^2}{16a^2} + \frac{7pq}{2} + 49a^2 - 48a^2 - \frac{3(p+q)^2}{4} &= 0 \\
p^2q^2 + 8a^2 \cdot 7pq + 16a^4 - 3(p+q)^2 \cdot 4a^2 &= 0 \\
p^2q^2 + 56pqa^2 + 16a^4 - 12p^2a^2 - 12q^2a^2 - 24pqa^2 &= 0 \\
p^2q^2 + 32pqa^2 + 16a^4 - 12p^2a^2 - 12q^2a^2 &= 0 \\
p^2q^2 + 32pqa^2 + 16a^4 - 3\left(\tan^2\theta p^2q^2 + 8(\tan^2\theta+1)a^2pq + 16\tan^2\theta a^4\right) &= 0 \\
(1-3\tan^2\theta)p^2q^2 + 8(1-3\tan^2\theta)pqa^2 + 16(1-3\tan^2\theta)a^4 &= 0 \\
(1-3\tan^2\theta)\left[p^2q^2 + 8pqa^2 + 16a^4\right] &= 0 \\
(1-3\tan^2\theta)(pq+4a^2)^2 &= 0.
\end{aligned}
$$

Hence, either $pq + 4a^2 = 0$, or $1 - 3\tan^2\theta = 0$. The former cannot always the case. Therefore, $1 - 3\tan^2\theta = 0$, which gives $\tan\theta = \pm\frac{\sqrt{3}}{3}$.

Since θ is acute, we have $\tan\theta = \frac{\sqrt{3}}{3}$, and hence $\theta = \frac{\pi}{6}$ is the acute angle between the two tangents.

2024.3 Question 5

1. Let

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \mathbf{N} = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

and hence we have

$$\operatorname{tr} \mathbf{M} = a + d, \operatorname{tr} \mathbf{N} = e + h.$$

Notice that

$$\mathbf{MN} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}, \mathbf{NM} = \begin{pmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{pmatrix},$$

which means

$$\operatorname{tr}(\mathbf{MN}) = ae + bg + cf + dh, \operatorname{tr}(\mathbf{NM}) = ae + cf + bg + dh,$$

and hence $\operatorname{tr}(\mathbf{MN}) = \operatorname{tr}(\mathbf{NM})$ as desired.

We also have

$$\mathbf{M} + \mathbf{N} = \begin{pmatrix} a + e & b + f \\ c + g & d + h \end{pmatrix},$$

meaning $\operatorname{tr}(\mathbf{M} + \mathbf{N}) = a + e + d + h = (a + d) + (e + h) = \operatorname{tr} \mathbf{M} + \operatorname{tr} \mathbf{N}$.

2. We have $\det \mathbf{M} = ad - bc$, and hence

$$\frac{\mathrm{d}}{\mathrm{d}t} \det \mathbf{M} = \dot{a}d + a\dot{d} - \dot{b}c - b\dot{c}.$$

Hence,

$$\text{LHS} = \frac{1}{ad - bd} \left(\dot{a}d + a\dot{d} - \dot{b}c - b\dot{c} \right).$$

On the other hand,

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \begin{pmatrix} \dot{a} & \dot{b} \\ \dot{c} & \dot{d} \end{pmatrix}, \mathbf{M}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and hence

$$\begin{aligned} \mathbf{M}^{-1} \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} &= \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \dot{a} & \dot{b} \\ \dot{c} & \dot{d} \end{pmatrix} \\ &= \frac{1}{ad - bc} \begin{pmatrix} \dot{a}d - b\dot{c} & \dot{b}d - b\dot{d} \\ -\dot{a}c + a\dot{c} & -\dot{b}c + a\dot{d} \end{pmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} \text{RHS} &= \operatorname{tr} \left(\mathbf{M}^{-1} \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} \right) \\ &= \frac{1}{ad - bc} \left(\dot{a}d - b\dot{c} - \dot{b}c + a\dot{d} \right) \\ &= \frac{1}{ad - bc} \left(\dot{a}d + a\dot{d} - b\dot{c} - \dot{b}c \right) \\ &= \text{LHS}, \end{aligned}$$

as desired.

3. $\det \mathbf{M} \neq 0$ since $\mathbf{M}$ is non-singular, and hence left-multiplying by $\mathbf{M}^{-1}$ on both sides gives us

$$\mathbf{M}^{-1} \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \mathbf{N} - \mathbf{M}^{-1}\mathbf{NM}.$$

Taking trace on both sides, we have

$$\begin{aligned}\frac{1}{\det \mathbf{M}} \frac{\mathrm{d}}{\mathrm{d}t} \det \mathbf{M} &= \operatorname{tr}\left(\mathbf{M}^{-1} \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t}\right) \\ &= \operatorname{tr}\left(\mathbf{N} - \mathbf{M}^{-1}\mathbf{N}\mathbf{M}\right) \\ &= \operatorname{tr}\mathbf{N} - \operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{N}\mathbf{M}\right) \\ &= \operatorname{tr}\mathbf{N} - \operatorname{tr}\left(\left(\mathbf{M}^{-1}\mathbf{N}\right)\mathbf{M}\right) \\ &= \operatorname{tr}\mathbf{N} - \operatorname{tr}\left(\mathbf{M}\left(\mathbf{M}^{-1}\mathbf{N}\right)\right) \\ &= \operatorname{tr}\mathbf{N} - \operatorname{tr}\left(\left(\mathbf{M}\mathbf{M}^{-1}\right)\mathbf{N}\right) \\ &= \operatorname{tr}\mathbf{N} - \operatorname{tr}\left(\mathbf{I}\mathbf{N}\right) \\ &= \operatorname{tr}\mathbf{N} - \operatorname{tr}\mathbf{N} \\ &= 0.\end{aligned}$$

Hence, $\frac{\mathrm{d}}{\mathrm{d}t} \det \mathbf{M} = 0$, which means $\det \mathbf{M}$ is a constant independent of t.
Directly taking trace on both sides, we have

$$\begin{aligned}\operatorname{tr} \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} &= \operatorname{tr}(\mathbf{MN} - \mathbf{NM}) \\ &= \operatorname{tr}(\mathbf{MN}) - \operatorname{tr}(\mathbf{NM}) \\ &= 0,\end{aligned}$$

and note

$$\operatorname{tr} \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{tr}\mathbf{M},$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{tr}\mathbf{M} = 0,$$

meaning $\operatorname{tr}\mathbf{M}$ is a constant independent of t.
Notice that

$$\operatorname{tr}\left(\mathbf{M}^2\right) = \operatorname{tr}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = a^2 + bc + bc + d^2 = a^2 + 2bc + d^2.$$

Since $\operatorname{tr}\mathbf{M}$ and $\det\mathbf{M}$ are both independent of t, we must have

$$\begin{aligned}(\operatorname{tr}\mathbf{M})^2 - 2\det\mathbf{M} &= (a+d)^2 - 2(ad - bc) \\ &= a^2 + 2ad + d^2 - 2ad + 2bc \\ &= a^2 + 2bc + d^2 \\ &= \operatorname{tr}\left(\mathbf{M}^2\right)\end{aligned}$$

is independent of t as well.
Let

$$\mathbf{M} = \begin{pmatrix} A + x & b \\ c & D - x \end{pmatrix},$$

the diagonal ones being so since the trace is independent of t. Here, x is a function of t.
By differentiating,

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \begin{pmatrix} \dot{x} & \dot{b} \\ \dot{c} & -\dot{x} \end{pmatrix},$$

and the right-hand side satisfies

$$\begin{aligned}\mathbf{MN} - \mathbf{NM} &= \begin{pmatrix} A + x & b \\ c & D - x \end{pmatrix}\begin{pmatrix} t & t \\ & t \end{pmatrix} - \begin{pmatrix} t & t \\ & t \end{pmatrix}\begin{pmatrix} A + x & b \\ c & D - x \end{pmatrix} \\ &= \begin{pmatrix} t(A + x) & (A + x)t + bt \\ ct & ct + (D - x)t \end{pmatrix} - \begin{pmatrix} t(A + x) + ct & bt + t(D - x) \\ ct & t(D - x) \end{pmatrix} \\ &= \begin{pmatrix} -ct & (A - D + 2x)t \\ 0 & ct. \end{pmatrix}\end{aligned}$$

Comparing the components, we see that $\dot{c} = 0$, meaning that c is a constant: $c = C$.
Hence, $\dot{x} = -Ct$, which solves to $x = -\frac{Ct^2}{2}$, since $x = 0$ when $t = 0$.
This means

$$\dot{b} = (A - D + 2x)t = (A - D - Ct^2)t,$$

and hence

$$b = \frac{(A-D)t^2}{2} - \frac{Ct^4}{4} + B$$

since $b = B$ when $t = 0$.
Hence,

$$\mathbf{M} = \begin{pmatrix} A - Ct^2/2 & (A-D)t^2/2 - Ct^4/4 \\ C & D + Ct^2/2 \end{pmatrix}$$

is the solution given the conditions.

4. By rearranging, we have

$$\mathbf{N} = \mathbf{M}^{-1} \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t}.$$

Hence, let

$$\mathbf{M} = \begin{pmatrix} 1 + e^t & \\ & 1 - e^t \end{pmatrix},$$

we have

$$\operatorname{tr} \mathbf{M} = 2$$

which is non-zero and independent of t.
Hence,

$$\mathbf{M}^{-1} = \frac{1}{1 - e^{2t}} \begin{pmatrix} 1 - e^t & \\ & 1 + e^t \end{pmatrix}, \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \begin{pmatrix} e^t & \\ & -e^t \end{pmatrix},$$

so

$$\begin{aligned} \mathbf{N} &= \frac{1}{1 - e^{2t}} \begin{pmatrix} 1 - e^t & \\ & 1 + e^t \end{pmatrix} \begin{pmatrix} e^t & \\ & -e^t \end{pmatrix} \\ &= \frac{1}{1 - e^{2t}} \begin{pmatrix} e^t(1 - e^t) & \\ & -e^t(1 + e^t) \end{pmatrix}, \end{aligned}$$

which gives

$$\operatorname{tr} \mathbf{N} = \frac{e^{2t}}{e^{2t} - 1}$$

which is clearly non-zero.

2024.3 Question 6

1. (a) We have

$$\begin{aligned}\frac{\mathrm{d}x-y}{\mathrm{d}t} &= \frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}y}{\mathrm{d}t} \\ &= (-x+3y+u)-(x+y+u) \\ &= -2x+2y \\ &= -2(x-y).\end{aligned}$$

This is a differential equation for $x-y$ in terms of t, and hence it solves to

$$x-y = Ae^{-2t}.$$

If $x=y=0$ for some $t>0$, then it must be the case that $A=0$, giving $x-y=0$, and $x=y$. Therefore, for $t=0$, we must also necessarily have $x_0=y_0$.

(b) Given that $x_0=y_0$, we must have $x=y$ for all $t>0$. Hence,

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -x+3x+u \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= 2x+u \\ \frac{\mathrm{d}x}{2x+u} &= \mathrm{d}t \\ \ln|2x+u| &= 2t+C \\ 2x+u &= Ae^{2t}.\end{aligned}$$

Since at $t=0$, $x=x_0$, we must have $A=2x_0+u$, and hence

$$2x+u=(2x_0+u)e^{2t},$$

and rearranging gives

$$u=\frac{2(x_0e^{2t}-x)}{1-e^{2t}}.$$

The particle is at origin at time $t=T>0$, and hence $x=y=0$ for $t=T$, and hence

$$u=\frac{2x_0e^{2T}}{1-e^{2T}}.$$

This ensures the particle is at origin as well since this ensures the particle is at $x=0$ for $t=T$, and $y=x$ so $y=0$ as well.

2. (a) Consider $\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}z}{\mathrm{d}t}-2\frac{\mathrm{d}y}{\mathrm{d}t}$, and we have

$$\begin{aligned}\frac{\mathrm{d}x+z-2y}{\mathrm{d}t} &= \frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}z}{\mathrm{d}t}-2\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= (4y-5z+u)+(x-2y+u)-2(x-2z+u) \\ &= 4y-5z+u+x-2y+u-2x+4z-2u \\ &= -x-z+2y,\end{aligned}$$

and hence

$$x+z-2y=Ae^{-t}.$$

Since the particle is at the origin at some time $t>0$, we must have $A=0$, and hence

$$x+z-2y=0,$$

which means $y=\frac{x+z}{2}$ for all time t.
At time $t=0$, $y_0=\frac{x_0+z_0}{2}$, and so y_0 is the mean of x_0 and z_0.

(b) Since $2y = x + z$, we must have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2(x+z) - 5z + u = 2x - 3z + u,$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = x - (x+z) + u = -z + u.$$

Hence, considering $\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}z}{\mathrm{d}t}$, we have

$$\begin{aligned}\frac{\mathrm{d}x-z}{\mathrm{d}t} &= \frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}z}{\mathrm{d}t} \\ &= (2x - 3z + u) - (-z + u) \\ &= 2(x-z),\end{aligned}$$

which gives

$$x - z = Ae^{2t}.$$

Since the particle is at the origin for some $t > 0$, we must have $A = 0$. This means $x = z$ for all t, and further we have $x = y = z$ for all t since $2y = x + z$.

At $t = 0$, this means $x_0 = y_0 = z_0$ as desired.

(c) Given that $x_0 = y_0 = z_0$, all previous parts still apply, since the boundary condition of $2y = x + z$ and $x = z$ holds for $t = 0$. Hence, $x = y = z$ for all t, and

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -x + u \\ \frac{\mathrm{d}x}{x-u} &= -\,\mathrm{d}t \\ \ln|x-u| &= -t + C \\ x - u &= Ae^{-t}.\end{aligned}$$

At $t = 0$, $x = x_0$, we must have $A = x_0 - u$, and hence

$$x - u = (x_0 - u)e^{-t},$$

and rearranging gives

$$u = \frac{x_0 e^{-t} - x}{1 - e^{-t}}.$$

The particle is at origin at a time $t = T > 0$, and hence $x = y = z = 0$ for $t = T$, and hence

$$u = \frac{x_0 e^{-T}}{1 - e^{-T}} = \frac{x_0}{1 + e^{T}}.$$

This ensures the particle is at origin as well since this ensures the particle is at $x = 0$ for $t = T$, and $x = y = z$, so $y = z = 0$ as well.

2024.3 Question 7

1. For the left inequality, $f(n) > 0$ since $f(n) > \frac{1}{n+1} > 0$.

   For the right inequality, we notice that

$$
\begin{aligned}
f(n) &= \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots \\
&< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots \\
&= \frac{1}{n+1} \cdot \frac{1}{1 - \frac{1}{n+1}} \\
&= \frac{1}{(n+1) - 1} \\
&= \frac{1}{n}.
\end{aligned}
$$

   Hence,

$$0 < f(n) < \frac{1}{n}.$$

2. For the left inequality, by grouping consecutive terms, we have

$$
\begin{aligned}
g(n) &= \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} \\
&\quad + \frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)} + \cdots \\
&= \left(\frac{1}{n+1} - \frac{1}{(n+1)(n+2)} \right) \\
&\quad + \left(\frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)} \right) + \cdots \\
&> \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \\
&\quad + \left(\frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)} \right) + \cdots \\
&= 0 + 0 + \cdots \\
&= 0,
\end{aligned}
$$

   using the inequality

$$\frac{1}{(n+1)\cdots(n+k)} > \frac{1}{(n+1)\cdots(n+k)(n+k+1)}.$$

For the right inequality, by grouping consecutive after the first one, we have

$$\begin{aligned}
g(n) &= \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} \\
&\quad - \frac{1}{(n+1)(n+2)(n+3)(n+4)} + \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)} - \cdots \\
&= \frac{1}{n+1} - \left(\frac{1}{(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)}\right) \\
&\quad - \left(\frac{1}{(n+1)(n+2)(n+3)(n+4)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)}\right) - \cdots \\
&< \frac{1}{n+1} - \left(\frac{1}{(n+1)(n+2)} - \frac{1}{(n+1)(n+2)}\right) \\
&\quad - \left(\frac{1}{(n+1)(n+2)(n+3)(n+4)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)}\right) - \cdots \\
&= \frac{1}{n+1} - 0 - 0 - \cdots \\
&= \frac{1}{n+1},
\end{aligned}$$

using the inequality

$$\frac{1}{(n+1)\cdots(n+k-1)(n+k)} < \frac{1}{(n+1)\cdots(n+k-1)}.$$

Hence,

$$0 < g(n) < \frac{1}{n+1}.$$

3. The infinite series for e is given by

$$e = \sum_{t=0}^{\infty} \frac{1}{t!},$$

and notice that

$$f(n) = \sum_{t=1}^{\infty} \frac{n!}{(n+t)!} = n! \sum_{t=1}^{\infty} \frac{1}{(n+t)!}.$$

Hence,

$$\begin{aligned}
(2n)!e - f(2n) &= (2n)! \sum_{t=0}^{\infty} \frac{1}{t!} - (2n)! \sum_{t=1}^{\infty} \frac{1}{(2n+t)!} \\
&= (2n)! \left(\sum_{t=0}^{\infty} \frac{1}{t!} - \sum_{t=2n+1}^{\infty} \frac{1}{t!}\right) \\
&= (2n)! \sum_{t=0}^{2n} \frac{1}{t!} \\
&= \sum_{t=0}^{2n} \frac{(2n)!}{t!}.
\end{aligned}$$

Since $t \leq 2n$, the terms in the sum represents the number of ways to arrange $(2n-t)$ items out of $2n$ items, which must be integers. Hence, the sum is an integer as well.

Similarly, the infinite series for e^{-1} is given by

$$e^{-1} = \sum_{t=0}^{\infty} \frac{(-1)^t}{t!},$$

and notice that

$$g(n) = -\sum_{t=1}^{\infty} \frac{(-1)^t n!}{(n+t)!} = -n! \sum_{t=1}^{\infty} \frac{(-1)^t}{(n+t)!}.$$

Hence,

$$\begin{aligned}\frac{(2n)!}{e} + g(2n) &= (2n)! \sum_{t=0}^{\infty} \frac{(-1)^t}{t!} - (2n)! \sum_{t=1}^{\infty} \frac{(-1)^t}{(n+t)!} \\ &= (2n)! \left(\sum_{t=0}^{\infty} \frac{(-1)^t}{t!} - \sum_{t=2n+1}^{\infty} \frac{(-1)^t}{t!} \right) \\ &= (2n)! \sum_{t=0}^{2n} \frac{(-1)^t}{t!} \\ &= \sum_{t=0}^{2n} \frac{(-1)^t (2n)!}{t!},\end{aligned}$$

and by the same argument, since $t \leq 2n$, this must be an integer as well.

4. By the previous part, let $a(n) = f(2n) - (2n)!e$, and $b(n) = g(2n) + \frac{(2n)!}{e}$, we must have that $a, b : \mathbb{N} \to \mathbb{Z}$ since they are integers.

   Using this notation,

$$\begin{aligned}qf(2n) + pg(2n) &= qa(2n) + qe(2n)! + pb(2n) - \frac{p}{e}(2n)! \\ &= qa(2n) + pb(2n) + \left(qe - \frac{p}{e}\right)(2n)! \\ &= qa(2n) + pb(2n)\end{aligned}$$

   must be an integer, since $p, q, a(2n), b(2n)$ are all integers.

5. Assume B.W.O.C. that e^2 is irrational. Then there exists natural numbers p, q such that

$$e^2 = \frac{p}{q} \iff qe = \frac{p}{e}.$$

   Since $e^2 > 1$, $p > q$.

   On one hand, we have $qf(2n) + pg(2n) > 0$.

   On the other hand, let $n = p$,

$$\begin{aligned}qf(2n) + pg(2n) &< q \cdot \frac{1}{2p} + p \cdot \frac{1}{2p+1} \\ &< q \cdot \frac{1}{2p} + p \cdot \frac{1}{2p} \\ &= \frac{p+q}{2p} \\ &< \frac{2p}{2p} \\ &= 1.\end{aligned}$$

   This means

$$0 < qf(2p) + pg(2p) < 1.$$

   But by the previous part, $qf(2n) + pg(2n)$ is an integer for all positive integer n, and $n = p$ is a positive integer. This leads to a contradiction.

   Hence, such p and q does not exist, meaning e^2 is not rational, hence e^2 is irrational.

2024.3 Question 8

1. $(y-x+3)(y+x-5)=0$ if and only if $y-x+3=0$, or $y+x-5=0$. In the first case, $y=x-3$, representing a straight line with gradient 1, and in the second case, $y=-x+5$, representing a straight line with gradient -1.

   The equation represents a pair of straight lines with gradients 1 and -1 if and only if it could be factorised into the form $(y-x+a)(y+x-b)$.

$$\begin{aligned}(y-x+a)(y+x+b) &= y^2+xy+by-xy-x^2-bx+ay+ax+ab\\ &= y^2-x^2+(a+b)y+(a-b)x+ab,\end{aligned}$$

   and $p=a+b, q=a-b, r=ab$.

   On one hand, if it could be factorised into this form, we have

$$p^2-q^2=(a+b)^2-(a-b)^2=a^2+2ab+b^2-a^2+2ab-b^2=4ab=4r.$$

   On the other hand, let $a=\frac{p+q}{2}$, $b=\frac{p-q}{2}$, and we have

$$a+b=p, a-b=q, ab=\frac{p+q}{2}\frac{p-q}{2}=\frac{p^2-q^2}{4}=\frac{4r}{4}=r.$$

   This shows that this is a necessary and sufficient condition, which finishes our proof.

2. Since the point (x,y) lies on C_1, we must have $y=x^2$, and $y-x^2=0$.

   Since it lies on C_2, we must have $x=y^2+2sy+s(s+1)$, and $y^2+2sy+s(s+1)-x$.

   Hence,

$$\begin{aligned}\text{LHS} &= y^2+2sy+s(s+1)-x+k(y-x^2)\\ &= 0+k\cdot 0\\ &= 0\\ &= \text{RHS}\end{aligned}$$

   for any real number k.

   Let $k=1$, by rearranging, we have

$$y^2-x^2+(2s+1)y-x+s(s+1)=0.$$

   We notice that

$$\begin{aligned}(2s+1)^2-(-1)^2 &= 4s^2+4s+1-1\\ &= 4s^2+4s\\ &= 4s(s+1),\end{aligned}$$

   which means that this represents a pair of straight lines with gradients 1 and -1. The four points of intersection must lie on them.

3. By part (ii), we have $a=\frac{(2s+1)-1}{2}=s$, and $b=\frac{(2s+1)-(-1)}{2}=s+1$. This means

$$(y-x+s)(y+x+s+1)=0,$$

   and the lines are $y=x-s$ and $y=-x-s-1$.

   Since a straight line may at most meet a polynomial twice, we must have $y=x-s$ meets $y=x^2$ at two distinct point, and $y=-x-s-1$ meets $y=x^2$ at two distinct points as well.

   $x^2=x-s \iff x^2-x+s=0$, and hence $1-4s>0$, which shows that $s<\frac{1}{4}$.

   $x^2--x-s-1 \iff x^2+x+(s+1)=0$, and hence $1-4(s+1)>0$, which shows that $s<-\frac{3}{4}$.

   Hence, $s<-\frac{3}{4}$.

4. The lines are $y = x - s$ and $y = -x - s - 1$. Since $s < -\frac{3}{4}$, both lines intersect $y = x^2$ on precisely two points, since the discriminant for the quadratic is positive. Hence, we just have to show that none of those four points are the same.

   This could only be the case of the intersection of the intersection of the two lines, which is $\left(-\frac{1}{2}, -\frac{2s+1}{2}\right)$. This lies on $y = x^2$ if and only if

$$-\frac{2s+1}{2} = \left(-\frac{1}{2}\right)^2 \iff -s - \frac{1}{2} = \frac{1}{4} \iff s = -\frac{3}{4}$$

   which is not the case here.

   Hence, C_1 and C_2 must intersect at four distinct points.

2024.3 Question 11

1. We notice that

$$\begin{aligned}\text{LHS} &= r\binom{2n}{r}\\ &= r \cdot \frac{(2n)!}{r!(2n-r)!}\\ &= \frac{(2n)!}{(r-1)!(2n-r)!},\end{aligned}$$

   and

$$\begin{aligned}\text{RHS} &= (2n+1-r)\binom{2n}{2n+1-r}\\ &= (2n+1-r) \cdot \frac{(2n)!}{(r-1)!(2n+1-r)!}\\ &= \frac{(2n)!}{(r-1)!(2n-r)!}.\end{aligned}$$

   Hence,

$$r\binom{2n}{r} = (2n+1-r)\binom{2n}{2n+1-r}$$

   as desired.

   Summing this from $r = n+1$ to $2n$, we have

$$\begin{aligned}\sum_{r=n+1}^{2n} r\binom{2n}{r} &= \sum_{r=n+1}^{2n} (2n+1-r)\binom{2n}{2n+1-r}\\ &= \sum_{r=1}^{n} (2n+1-(2n+1-r))\binom{2n}{2n+1-(2n+1-r)}\\ &= \sum_{r=1}^{n} r\binom{2n}{r},\end{aligned}$$

   and hence

$$\begin{aligned}\sum_{r=0}^{2n} r\binom{2n}{r} &= \sum_{r=1}^{2n} r\binom{2n}{r}\\ &= \sum_{r=1}^{n} r\binom{2n}{r} + \sum_{r=n+1}^{2n} r\binom{2n}{r}\\ &= \sum_{r=n+1}^{2n} r\binom{2n}{r} + \sum_{r=n+1}^{2n} r\binom{2n}{r}\\ &= 2\sum_{r=n+1}^{2n} r\binom{2n}{r},\end{aligned}$$

   as desired.

2. For $n+1 \leq x \leq 2n$, we have

$$\mathrm{P}(X = x) = 2 \cdot \frac{\binom{2n}{x}}{2^{2n}}.$$

   For $x = n$, we have

$$\mathrm{P}(X = x) = \frac{\binom{2n}{n}}{2^{2n}}.$$

We have $n \leq X \leq 2n$, and hence

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{x=n}^{2n} x \, \mathrm{P}(X = x) \\
&= \frac{n\binom{2n}{n}}{2^{2n}} + \frac{2}{2^{2n}} \sum_{x=n+1}^{2n} x \binom{2n}{x} \\
&= \frac{n\binom{2n}{n}}{2^{2n}} + 2^{-2n} \sum_{r=0}^{2n} r \binom{2n}{r} \\
&= \frac{n\binom{2n}{n}}{2^{2n}} + 2^{-2n}(2n)2^{2n-1} \\
&= n + \frac{n\binom{2n}{n}}{2^{2n}} \\
&= n\left(1 + \frac{1}{2^{2n}}\binom{2n}{n}\right)
\end{aligned}$$

as desired.

3. First, we have that

$$\frac{1}{2^{2n}}\binom{2n}{n} > 0$$

for all positive integers n.

Taking the ratio of two consecutive terms, we have

$$\begin{aligned}
\frac{\frac{1}{2^{2n}}\binom{2n}{n}}{\frac{1}{2^{2(n+1)}}\binom{2(n+1)}{n+1}} &= \frac{2^{2n+2}\frac{(2n)!}{n!n!}}{2^{2n}\frac{(2n+2)!}{(n+1)!(n+1)!}} \\
&= 4 \cdot \frac{(n+1)^2}{(2n+2)(2n+1)}.
\end{aligned}$$

We have that the following are equivalent:

$$\begin{aligned}
\frac{1}{2^{2n}}\binom{2n}{n} &> \frac{1}{2^{2(n+1)}}\binom{2(n+1)}{n+1} \\
\frac{\frac{1}{2^{2n}}\binom{2n}{n}}{\frac{1}{2^{2(n+1)}}\binom{2(n+1)}{n+1}} &> 1 \\
\frac{4(n+1)^2}{(2n+2)(2n+1)} &> 1 \\
4n^2 + 8n + 4 &> 4n^2 + 6n + 2 \\
2n + 2 &> 0
\end{aligned}$$

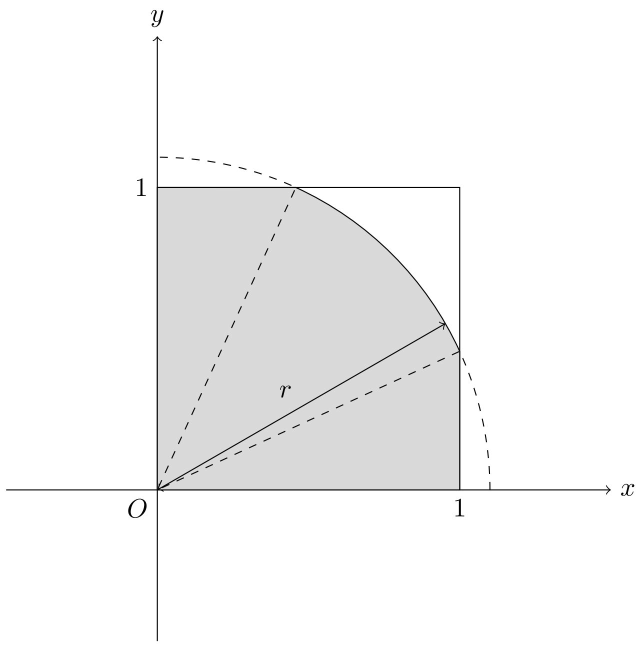
and this obviously true for all positive integers n.

This means that $\frac{1}{2^{2n}}\binom{2n}{n}$ decreases as n increases.

4. The winning is given by $X - n$, and hence the expected winnings per pound is $\frac{1}{2^{2n}}\binom{2n}{n}$. This is maximised when $n = 1$ which gives a value of $\frac{1}{2}$.

2024.3 Question 12

1. For $1 \leq r \leq \sqrt{2}$, the diagram looks as follows.



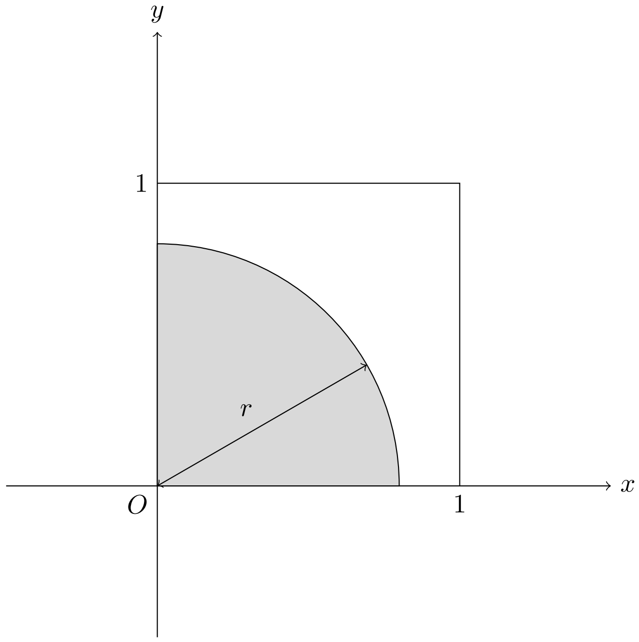
The angle between the (shallower) radius which just intersects the square and x axis is given by $\arccos \frac{1}{r}$, and so is the one steeper and the y-axis.

Hence, the cumulative distribution function is given by

$$\begin{aligned}
\mathrm{P}(R \leq r) &= \frac{\text{shaded area}}{1^2} \\
&= \text{shaded area} \\
&= \frac{1}{2} \cdot r^2 \cdot \left(\frac{\pi}{2} - 2 \arccos \frac{1}{r}\right) + 2 \cdot \frac{1}{2} \cdot 1 \cdot \sqrt{r^2 - 1} \\
&= \sqrt{r^2 - 1} + \frac{\pi r^2}{4} - r^2 \arccos \frac{1}{r},
\end{aligned}$$

as desired.

For $0 \leq r \leq 1$, the diagram is as follows.



Hence,

$$\mathrm{P}(R \leq r) = \text{shaded area} = \frac{\pi r^2}{4}.$$

Hence, the cumulative distribution function is given by

$$\mathrm{P}(R \leq r) = \begin{cases} 0, & r < 0, \\ \frac{\pi r^2}{4}, & 0 \leq r < 1, \\ \sqrt{r^2 - 1} + \frac{\pi r^2}{4} - r^2 \arccos \frac{1}{r}, & 1 \leq r < 2, \\ 1, & 2 \leq r. \end{cases}$$

2. Let f be the probability density function of R. Hence, by differentiating, for $0 \leq r \leq \sqrt{2}$, it is given by

$$\begin{aligned} f(r) &= \frac{\mathrm{d}}{\mathrm{d}r} \mathrm{P}(R \leq r) \\ &= \begin{cases} \frac{\pi r}{2}, & 0 \leq r \leq 1, \\ \frac{r}{\sqrt{r^2-1}} + \frac{\pi r}{2} - 2r \arccos \frac{1}{r} - \frac{1}{\sqrt{1-\left(\frac{1}{r}\right)^2}}, & 1 \leq r \leq \sqrt{2}, \end{cases} \\ &= \begin{cases} \frac{\pi r}{2}, & 0 \leq r \leq 1, \\ \frac{\pi r}{2} - 2r \arccos \frac{1}{r}, & 1 \leq r \leq \sqrt{2}. \end{cases} \end{aligned}$$

Hence, the expectation is given by

$$\begin{aligned}
\mathrm{E}(R) &= \int_0^1 r \cdot \frac{\pi r}{2} \, \mathrm{d}r + \int_1^{\sqrt{2}} r \cdot \left[\frac{\pi r}{2} - 2r \arccos \frac{1}{r}\right] \mathrm{d}r \\
&= \int_0^{\sqrt{2}} \frac{\pi r^2}{2} \, \mathrm{d}r - 2 \int_1^{\sqrt{2}} r^2 \arccos \frac{1}{r} \, \mathrm{d}r \\
&= \left[\frac{\pi r^3}{6}\right]_0^{\sqrt{2}} - \frac{2}{3} \int_1^{\sqrt{2}} \arccos \frac{1}{r} \, \mathrm{d}r^3 \\
&= \frac{2\sqrt{2}\pi}{6} - \frac{2}{3} \left[\arccos \frac{1}{r} \cdot r^3\right]_1^{\sqrt{2}} + \frac{2}{3} \int_1^{\sqrt{2}} r^3 \, \mathrm{d} \arccos \frac{1}{r} \\
&= \frac{\sqrt{2}\pi}{3} - \frac{2}{3} \cdot \arccos \frac{1}{\sqrt{2}} \cdot 2\sqrt{2} + \frac{2}{3} \cdot \arccos 1 \cdot 1 + \frac{2}{3} \cdot \int_1^{\sqrt{2}} r^3 \cdot \left(-\frac{1}{r^2}\right) \cdot \left(-\frac{1}{\sqrt{1 - \left(\frac{1}{r}\right)^2}}\right) \mathrm{d}r \\
&= \frac{\sqrt{2}\pi}{3} - \frac{2}{3} \cdot \frac{\pi}{4} \cdot 2\sqrt{2} + \frac{2}{3} \int_1^{\sqrt{2}} r \cdot \frac{r}{\sqrt{r^2 - 1}} \, \mathrm{d}r \\
&= \frac{\sqrt{2}\pi}{3} - \frac{\sqrt{2}\pi}{3} + \frac{2}{3} \int_1^{\sqrt{2}} \frac{r^2}{\sqrt{r^2 - 1}} \, \mathrm{d}r \\
&= \frac{2}{3} \int_1^{\sqrt{2}} \frac{r^2}{\sqrt{r^2 - 1}} \, \mathrm{d}r,
\end{aligned}$$

as desired.

3. To integrate this, we let $r = \cosh t$, and hence $\frac{\mathrm{d}r}{\mathrm{d}t} = \sinh t$. When $r = 1$, $t = 0$. When $r = \sqrt{2}$, $t = \ln\left(\sqrt{2} + \sqrt{\sqrt{2}^2 - 1}\right) = \ln(\sqrt{2} + 1)$.

   Hence,

$$\begin{aligned}
\mathrm{E}(R) &= \frac{2}{3} \int_1^{\sqrt{2}} \frac{r^2}{\sqrt{r^2 - 1}} \, \mathrm{d}r \\
&= \frac{2}{3} \int_0^{\ln(\sqrt{2}+1)} \frac{\cosh^2 t}{\sinh t} \cdot \sinh t \, \mathrm{d}t \\
&= \frac{2}{3} \int_0^{\ln(\sqrt{2}+1)} \cosh^2 t \, \mathrm{d}t \\
&= \frac{2}{3} \int_0^{\ln(\sqrt{2}+1)} \frac{e^{2t} + e^{-2t} + 2}{4} \, \mathrm{d}t \\
&= \frac{1}{2} \left[e^{2t} - e^{-2t}\right]_0^{\ln(\sqrt{2}+1)} + \frac{1}{3} [t]_0^{\ln(\sqrt{2}+1)} \\
&= \frac{1}{12} \cdot \left[(\sqrt{2} + 1)^2 - (\sqrt{2} + 1)^{-2} - e^{2 \cdot 0} + e^{-2 \cdot 0}\right] + \frac{1}{3} \cdot \left(\ln(\sqrt{2} + 1) - 0\right) \\
&= \frac{1}{2} \left[2 + 1 + 2\sqrt{2} - (\sqrt{2} - 1)^2\right] + \frac{1}{3} \ln(\sqrt{2} + 1) \\
&= \frac{1}{2} \cdot 4\sqrt{2} + \frac{1}{3} \ln(\sqrt{2} + 1) \\
&= \frac{1}{3} \left(\sqrt{2} + \ln\left(\sqrt{2} + 1\right)\right),
\end{aligned}$$

as desired.