Year 2019

2019 Paper 3

2019.3 Question 1

1. When $k = 1$,

$$\dot{x} = -x - y, \dot{y} = x - y.$$

Hence,

$$\begin{aligned}\ddot{x} &= -\dot{x} - \dot{y} \\ &= -\dot{x} - (x - y) \\ &= -\dot{x} - x + y \\ &= -\dot{x} - x + (-x - \dot{x}) \\ &= -2\dot{x} - 2x,\end{aligned}$$

and this gives

$$\ddot{x} + 2\dot{x} + 2x = 0.$$

The auxiliary equation to this differential equation is

$$\lambda^2 + 2\lambda + 2 = 0,$$

which solves to

$$\lambda = -1 \pm i.$$

The general solution for x is hence

$$x(t) = \exp(-t) \left(A \sin t + B \cos t\right).$$

This means

$$\begin{aligned}\dot{x}(t) &= -\exp(-t) \left(A \sin t + B \cos t\right) + \exp(-t) \left(A \cos t - B \sin t\right) \\ &= -x(t) + \exp(-t) \left(A \cos t - B \sin t\right),\end{aligned}$$

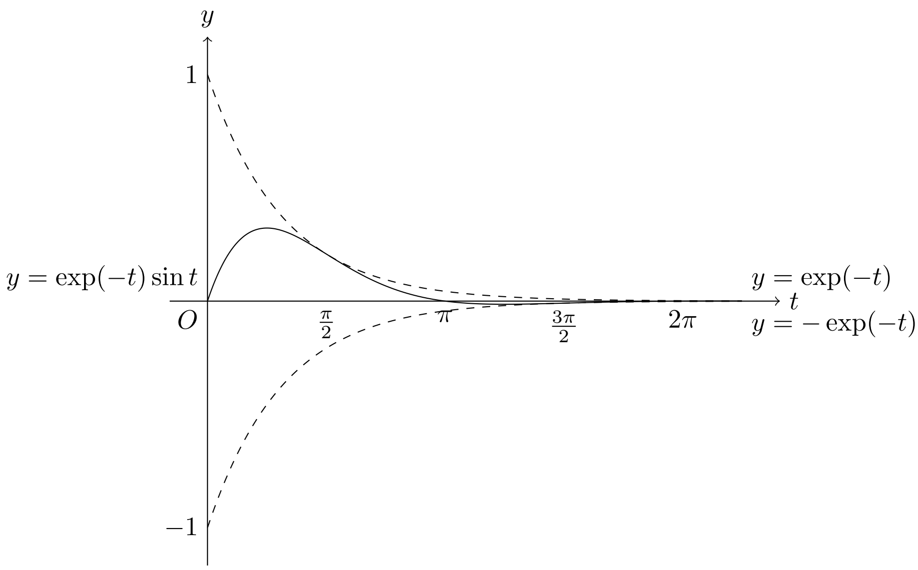
and hence

$$y(t) = -\exp(-t) \left(A \cos t - B \sin t\right) = \exp(-t) \left(B \sin t - A \cos t\right).$$

When $t = 0$, $x = x(0) = B = 1$, $y = y(0) = -A = 0$. Hence,

$$x(t) = \exp(-t) \cos t, y(t) = \exp(-t) \sin t.$$

The graph of y against t looks as follows:



y is greatest at the first stationary point of y, as shown in the graph. Note that

$$\dot{y} = x - y = \exp(-t)\left(\cos t - \sin t\right),$$

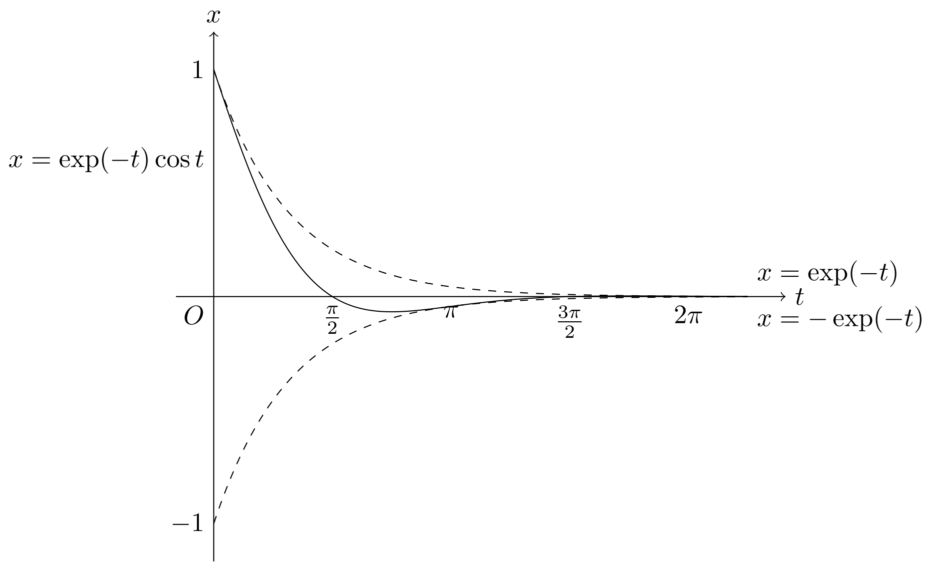
and hence

$$\dot{y} = 0 \iff \cos t = \sin t \iff \tan t = 1,$$

and the smallest positive solution to this is $t = \frac{\pi}{4}$. The coordinate of the point is hence

$$(x, y) = \left(\exp\left(-\frac{\pi}{4}\right) \cdot \frac{\sqrt{2}}{2}, \exp\left(-\frac{\pi}{4}\right) \cdot \frac{\sqrt{2}}{2}\right).$$

Similarly, the graph of x against t looks as follows:



x is smallest at the first stationary point of x, as shown in the graph. Note that

$$\dot{x} = -x - y = -\exp(-t)\left(\cos t + \sin t\right),$$

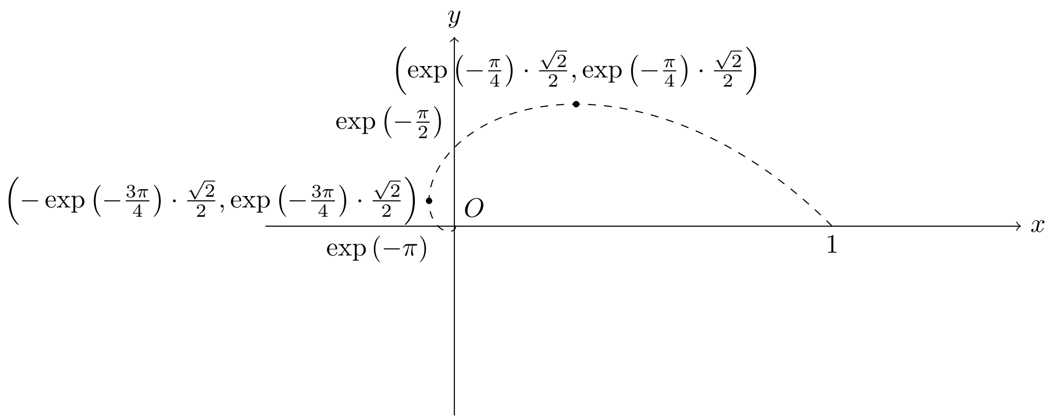
and hence

$$\dot{x} = 0 \iff \cos t = -\sin t \iff \tan t = -1,$$

and the smallest positive solution to this is $t = \frac{3\pi}{4}$. The coordinate of the point is hence

$$(x, y) = \left(-\exp\left(-\frac{3\pi}{4}\right) \cdot \frac{\sqrt{2}}{2}, \exp\left(-\frac{3\pi}{4}\right) \cdot \frac{\sqrt{2}}{2}\right).$$

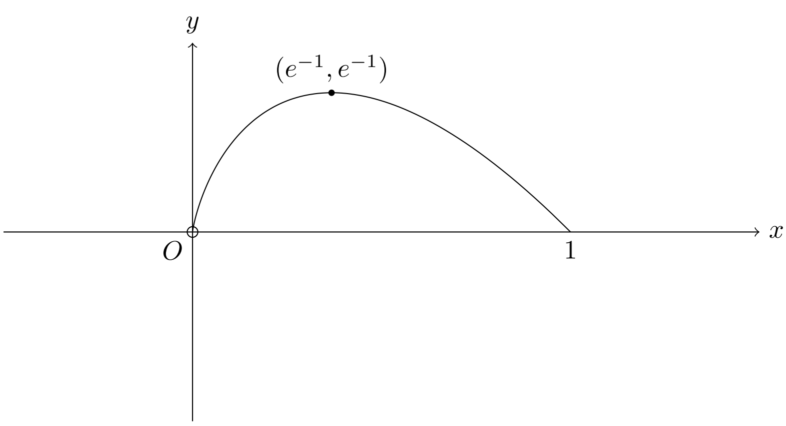
Without the $\exp(-t)$ factor, the x-y graph will simply be a circle, and with this factor, it will be a spiral with exponentially decreasing radius. This is the polar curve $r = \exp(-\theta)$. Hence, the x-y graph looks as follows.



2. Since $\dot{x} = -x$, we must have $x(t) = A\exp(-t)$, and since $x(0) = 1$, we have $A = 1$ and $x(t) = \exp(-t)$.

We have
$$\dot{y} = \exp(-t) - y,$$
and hence
$$\dot{y} + y = \exp(-t).$$
Multiplying both sides by $\exp(t)$, we have
$$e^t\dot{y} + e^t y = 1,$$
and hence
$$\frac{\mathrm{d}ye^t}{\mathrm{d}t} = 1,$$
which gives
$$ye^t = t + B,$$
and hence
$$y = \exp(-t)(t + B).$$
Since $y = 0$ when $t = 0$, we must have $B = 0$, and hence
$$y = t\exp(-t).$$
Note that
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \exp(-t) - t\exp(-t),$$
and hence $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ when $t = 1$, which is when
$$(x, y) = \left(e^{-1}, e^{-1}\right).$$
Note that
$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\exp(-t),$$
and hence $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$, which is when
$$(x, y) = (1, 0),$$
and the tangent to the curve at this point will be vertical.

Hence, the graph will look as follows:



2019.3 Question 2

1. Let $y = 0$, and we have
$$f(x+0) = f(x) = f(x)f(0),$$
so either $f(x) = 0$ or $f(0) = 1$ for all x.

   Assume, B.W.O.C., that $f(0) \neq 1$, then we must have $f(x) = 0$ for all x, which means $f'(x) = 0$, contradicting with $f'(0) = k \neq 0$.

   Hence, $f(0) = 1$.

   By definition of the derivative, we have
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \lim_{h \to 0} \frac{f(h) - 1}{h}, \end{aligned}$$
and letting $x = 0$, we also have
$$k = f'(0) = f(0) \lim_{h \to 0} \frac{f(h) - 1}{h} = \lim_{h \to 0} \frac{f(h) - 1}{h},$$
and hence
$$f'(x) = kf(x)$$
as desired.

   This differential equation solves to
$$f(x) = Ae^{kx},$$
and with the condition $f(0) = 1$, we have $A = 1$, and hence
$$f(x) = e^{kx}$$
for all x.

2. Let $y = 0$, and we have
$$g(x+0) = g(x) = \frac{g(x) + g(0)}{1 + g(x)g(0)}.$$
This means that
$$g(x) + g(x)^2 g(0) = g(x) + g(0),$$
which gives
$$g(0)\left[g(x)^2 - 1\right] = 0.$$
Since $|g(x)| < 1$ for all x, we must have $g(x)^2 - 1 < 0$, and hence $g(0) = 0$.

   By the definition of the derivative,
$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{\frac{g(x)+g(h)}{1+g(x)g(h)} - g(x)}{h} \\ &= \lim_{h \to 0} \frac{g(x) + g(h) - g(x) - g(x)^2 g(h)}{h(1 + g(x)g(h))} \\ &= \lim_{h \to 0} \frac{g(h)\left[1 - g(x)^2\right]}{h(1 + g(x)g(h))} \\ &= \left[1 - g(x)^2\right] \lim_{h \to 0} \frac{g(h)}{h(1 + g(x)g(h))}. \end{aligned}$$

Considering the limit, we have

$$\begin{aligned}\lim_{h\to 0}\frac{g(h)}{h(1+g(x)g(h))} &= \lim_{h\to 0}\frac{g(h)/h}{1+g(x)g(h)}\\ &= \frac{\lim_{h\to 0}[g(h)/h]}{\lim_{h\to 0}[1+g(x)g(h)]}\\ &= \frac{\lim_{h\to 0}[g(h)/h]}{1}\\ &= \lim_{h\to 0}\frac{g(h)}{h},\end{aligned}$$

and hence

$$g'(x) = \left[1-g(x)^2\right]\lim_{h\to 0}\frac{g(h)}{h}.$$

Let $x=0$, and we have

$$k = g'(0) = 1\cdot\lim_{h\to 0}\frac{g(h)}{h},$$

hence giving the differential equation

$$g'(x) = k\left[1-g(x)^2\right].$$

This rearranges to give

$$\frac{\mathrm{d}g(x)}{1-g(x)^2} = k\,\mathrm{d}x,$$

and hence

$$\left[\frac{1}{1+g(x)}+\frac{1}{1-g(x)}\right]\mathrm{d}g(x) = 2k\,\mathrm{d}x,$$

which gives

$$\ln|1+g(x)| - \ln|1-g(x)| = 2kx + C.$$

Let $x=0$, we have $g(0)=0$, and hence $C=0$, and hence

$$\frac{1+g(x)}{1-g(x)} = \exp(2kx),$$

and hence

$$1+g(x) = \exp(2kx) - \exp(2kx)g(x),$$

which gives

$$g(x) = \frac{\exp(2kx)-1}{\exp(2kx)+1} = \frac{\exp(kx)-\exp(-kx)}{\exp(kx)+\exp(-kx)} = \tanh(kx).$$

2019.3 Question 3

1. Since L_1 is a line of invariant points, for each point $(x, y) \in L_1$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

and hence

$$ax + by = x, cx + dy = y.$$

Hence,

$$(1 - a)x = by, (1 - d)y = cx,$$

and hence

$$(1 - a)x(1 - d)y = bycx,$$

which simplifies to

$$[(a - 1)(d - 1) - bc]xy = 0.$$

If the line L_1 is the line $x = 0$, then $by = 0$ for all y and $dy = y$ for all y, giving $d = 1$ and $b = 0$. Hence, $(a - 1)(d - 1) - bc = 0$.

Similarly, if the line L_1 is the line $y = 0$, then $ax = x$ for all x and $cx = 0$ for all y, giving $a = 1$ and $c = 0$. Hence, $(a - 1)(d - 1) - bc = 0$.

Otherwise, there must be a point $(x, y) \in L_1$ such that $xy \neq 0$, which means $(a-1)(d-1)-bc = 0$.

Hence, in all cases, we must have $(a - 1)(d - 1) = bc$ as desired.

If L_1 does not pass through the origin, then $y = mx + k$ for some $k \neq 0$, or $x = k$ for some $k \neq 0$.

In the first case, we have

$$ax + b(mx + k) = x,$$

and hence

$$(a + bm - 1)x + bk = 0$$

for all x, meaning $a + bm - 1 = 0$ and $bk = 0$.

Similarly,

$$cx + d(mx + k) = mx + k,$$

and hence

$$(c + dm - m)x + (d - 1)k = 0$$

for all x, meaning $c + dm - m = 0$ and $(d - 1)k = 0$.

Since $k \neq 0$, $bk = 0$ and $(d - 1)k = 0$ implies $b = 0$ and $d = 1$ respectively. Putting those back into the first corresponding equations, this solves to $a = 1$ and $c = 0$, which means

$$\mathbf{A} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \mathbf{I}_2.$$

In the second case where $x = k$ for some $k \neq 0$, we have

$$ak + by = k,$$

and hence

$$by + (a - 1)k = 0$$

for all y, meaning $b = 0$ and $(a - 1)k = 0$.

Similarly,

$$ck + dy = y,$$

and hence

$$(d - 1)y + ck = 0$$

for all y, meaning $d - 1 = 0$ and $ck = 0$.

Since $k \neq 0$, $(a-1)k = 0$ and $ck = 0$ implies $a = 1$ and $c = 0$ respectively. Hence,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}_2.$$

Therefore, L_1 not passing through the origin must imply that $\mathbf{A}$ is precisely the 2 by 2 identity matrix.

2. If (x, y) is an invariant point, we have

$$(a-1)x + by = 0, cx + (d-1)y = 0.$$

If $b = 0$, then $(a-1)(d-1) = bc = 0$, and hence $a = 1$ or $d = 1$.

In the case where $a = 1$, the first equation is trivially true, and the second equation simplifies to

$$cx + (d-1)y = 0,$$

and hence the line $L : cx + (d-1)y = 0$ is a line of invariant points.

In the case where $d = 1$, the original equation simplifies to

$$(a-1)x = 0, cx = 0,$$

and hence the line $L : x = 0$ is a line of invariant points.

If $b \neq 0$, we want to show that all points on the line $L : (a-1)x + by = 0$ satisfy the second equation. We multiply $(d-1)$ on both sides of the equation, and hence

$$(a-1)(d-1)x + b(d-1)y = 0,$$

which is

$$bcx + b(d-1)y = 0.$$

Since $b \neq 0$, we divide b on both sides, giving

$$cx + (d-1)y = 0,$$

which is precisely the second equation. Hence, $L : (a-1)x + by = 0$ is a line of invariant points under this case.

3. We have $L_2 : y = mx + k$, $k \neq 0$, we therefore have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ mx + k \end{pmatrix} = \begin{pmatrix} X \\ mX + k \end{pmatrix},$$

and hence

$$ax + b(mx + k) = X, cx + d(mx + k) = mX + k.$$

Putting the first equation into the second one gives us

$$cx + d(mx + k) = m(ax + b(mx + k)) + k,$$

which simplifies to

$$(c + dm - am - bm^2)x + (dk - mbk - k) = 0,$$

which is

$$(bm^2 + (a-d)m - c)x + (mb - d + 1)k = 0.$$

Since this is true for all x and $k \neq 0$, we must have

$$bm^2 + (a-d)m - c = 0, bm - d + 1 = 0.$$

If $b = 0$, then

$$(a-d)m = c, d - 1 = 0,$$

and hence $d = 1$, $(a-1)m = c$, and

$$(a-1)(d-1) = 0, bc = 0,$$

which gives

$$(a-1)(d-1) = bc.$$

If $b \neq 0$, the second of those equations solve to

$$m = \frac{d-1}{b},$$

and putting this back into the first equation, we have

$$b \cdot \frac{(d-1)^2}{b^2} + \frac{(a-d)(d-1)}{b} - c = 0,$$

and multiplying both sides by b gives

$$(d-1)^2 + (a-d)(d-1) = bc,$$

and hence

$$(a-1)(d-1) = bc.$$

Therefore, in both cases, we have $(a-1)(d-1) = bc$, as desired.

2019.3 Question 4

1. We look at different cases depending on the value of n.

   - When $n = 1$, $P(x) = x - a_1$ has root a_1, and thus is reflective for all $a_1 \in \mathbb{R}$.
   - When $n = 2$, $P(x) = x^2 - a_1 x + a_2$ has root a_1, a_2, and hence by Vieta's Theorem,

     $$a_1 a_2 = a_2, a_1 + a_2 = a_1.$$

     This means $a_2 = 0$ and a_1 can take any real value, and hence

     $$P(x) = x^2 - a_1 x$$

     is reflective for $a_1 \in \mathbb{R}$.
   - When $n = 3$, $P(x) = x^3 - a_1 x^2 + a_2 x - a_3$ has root a_1, a_2, a_3, and hence by Vieta's Theorem,

     $$\begin{cases} a_1 a_2 a_3 = a_3, \\ a_1 a_2 + a_1 a_3 + a_2 a_3 = a_2, \\ a_1 + a_2 + a_3 = a_1. \end{cases}$$

     The final equation implies that $a_2 + a_3 = 0$, and hence with the second equation gives that $a_2 a_3 = a_2$, which means either $a_2 = a_3 = 0$, or $a_2 = -1, a_3 = 1$.
     When $a_2 = a_3 = 0$, a_1 can take any real value, and when $a_2 = -1, a_3 = 1$, we must have $a_1 = -1$.
     So the degree 3 reflective polynomials are

     $$P(x) = x^3 - a_1 x^2$$

     for all $a_1 \in \mathbb{R}$, and

     $$P(x) = x^3 + x^2 - x - 1.$$

2. By Vieta's Theorem, we have

   $$\sum_{i=1}^{n} a_i = a_1,$$

   and hence

   $$\sum_{i=2}^{n} a_i = 0.$$

   Squaring both sides gives

   $$\begin{aligned} 0 &= \left(\sum_{i=2}^{n} a_i \right)^2 \\ &= \sum_{i=2}^{n} a_i^2 + 2 \sum_{i=2}^{n-1} \sum_{j=i+1}^{n} a_i a_j. \end{aligned}$$

   By Vieta's Theorem, we also have

   $$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_i a_j = a_2,$$

and notice that

$$\begin{aligned}2a_2 &= 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i a_j \\ &= 2\sum_{j=2}^{n} a_1 a_j + 2\sum_{i=2}^{n-1}\sum_{j=i+1}^{n} a_i a_j \\ &= 2a_1\sum_{i=2}^{n} a_i + \left(-\sum_{i=2}^{n} a_i^2\right) \\ &= 2a_1 \cdot 0 - \sum_{i=2}^{n} a_i^2 \\ &= -\sum_{i=2}^{n} a_i^2,\end{aligned}$$

as desired.

For the final part, assume B.W.O.C. that $n > 3$. By rearrangement, we have

$$a_2^2 + 2a_2 + 1 = 1 - \sum_{i=3}^{n} a_i^2,$$

and the left-hand side is $(a_2+1)^2$ which is always non-negative. Hence,

$$\sum_{i=3}^{n} a_i^2 \le 1.$$

Since a_i are all integers, precisely one of the a_is for $3 \le i \le n$ is ± 1, and all the rest are 0. Since $a_n \neq 0$, we conclude that $a_n = \pm 1$, and $a_3 = \cdots = a_{n-1} = 0$.

But notice from Vieta's Theorem that

$$a_n = \prod_{i=1}^{n} a_i = 0$$

since a_3 must be 0, which leads to a contradiction.

Hence, we must have $n \le 3$.

3. The reflective polynomials for $n \le 3$ are

- $P(x) = x - a_1$ for $a_1 \in \mathbb{Z}$,
- $P(x) = x^2 - a_1 x$ for $a_1 \in \mathbb{Z}$,
- $P(x) = x^3 - a_1 x^2$ for $a_1 \in \mathbb{Z}$, and
- $P(x) = x^3 + x^2 - x - 1$.

For $n > 3$, we must have $a_n = 0$, and hence

$$\begin{aligned}P(x) &= x^n - a_1 x^{n-1} + a_2 x^{n-2} - \cdots + (-1)^{n-1} a_{n-1} x \\ &= x\left(x^{n-1} - a_2 x^{n-2} + a_2 x^{n-3} - \cdots + (-1)^{n-1} a_{n-1}\right).\end{aligned}$$

Let

$$Q(x) = x^{n-1} - a_2 x^{n-2} + a_2 x^{n-3} - \cdots + (-1)^{n-1} a_{n-1}$$

If $P(x)$ is reflective, then the roots to $P(x)$ are $a_1, a_2, \ldots, a_{n-1}, 0$, and hence the roots to $Q(x)$ are $a_1, a_2, \ldots, a_{n-1}$, which shows that $Q(x)$ is reflective as well.

This means that an integer-coefficient reflective polynomial with degree $n > 3$ must be x multiplied by another integer-coefficient reflective polynomial, and repeating this process, we can conclude it must be some power of x multiplied by some integer-coefficient reflective polynomial with degree $n \le 3$.

Hence, all integer-coefficient reflective polynomials are

- $P(x) = x^r(x - a_1)$ for $a_1 \in \mathbb{Z}$, $r \in \mathbb{Z}$, $r \geq 0$, and
- $P(x) = x^r(x^3 + x^2 - x - 1) = x^2(x+1)^2(x-1)$ for $r \in \mathbb{Z}$, $r \geq 0$.

2019.3 Question 5

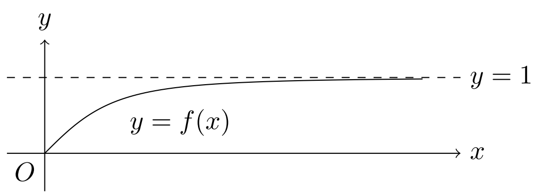
1. By quotient rule,

$$\begin{aligned} f'(x) &= \frac{\sqrt{x^2+p} - x \cdot \frac{1}{2} \cdot 2x \cdot \frac{1}{\sqrt{x^2+p}}}{x^2+p} \\ &= \frac{\sqrt{x^2+p} - \frac{x^2}{\sqrt{x^2+p}}}{x^2+p} \\ &= \frac{p}{(x^2+p)\sqrt{x^2+p}}. \end{aligned}$$

This gives

$$0 < f'(x) \leq \frac{1}{\sqrt{p}},$$

with the equal sign taking if and only if $x = 0$.
$\lim_{x\to\infty} f(x) = 1$, so $y = 1$ is a horizontal asymptote to the function.
Hence, the graph looks as follows:



2. Since $y = \frac{cx}{\sqrt{x^2+p}} = cf(x)$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = cf'(x) = \frac{cp}{\left(\sqrt{x^2+p}\right)^3},$$

and hence

$$\mathrm{d}y = \frac{cp}{\left(\sqrt{x^2+p}\right)^3}\,\mathrm{d}x.$$

The integral can therefore be simplified as

$$\begin{aligned} I &= \int \frac{\mathrm{d}y}{(b^2-y^2)\sqrt{c^2-y^2}} \\ &= \int \frac{1}{\left(b^2 - \frac{c^2x^2}{x^2+p}\right)\sqrt{c^2 - \frac{c^2x^2}{x^2+p}}} \cdot \frac{cp}{\left(\sqrt{x^2+p}\right)^3}\,\mathrm{d}x \\ &= \int \frac{cp\,\mathrm{d}x}{(b^2(x^2+p) - c^2x^2)\sqrt{c^2(x^2+p) - c^2x^2}} \\ &= \int \frac{cp\,\mathrm{d}x}{[(b^2-c^2)x^2 + b^2p]\sqrt{c^2p}} \\ &= \int \frac{\sqrt{p}\,\mathrm{d}x}{b^2p + (b^2-c^2)x^2}. \end{aligned}$$

Let $p = 1$, and we have

$$I = \int \frac{\mathrm{d}x}{b^2 + (b^2-c^2)x^2}$$

as desired.

Hence,

$$\begin{aligned} I &= \int \frac{\mathrm{d}x}{b^2 + (b^2 - c^2)x^2} \\ &= \frac{1}{b^2 - c^2} \int \frac{\mathrm{d}x}{\left(\frac{b}{\sqrt{b^2-c^2}}\right)^2 + x^2} \\ &= \frac{1}{b^2 - c^2} \cdot \frac{\sqrt{b^2 - c^2}}{b} \arctan \frac{\sqrt{b^2 - c^2}x}{b} + C \\ &= \frac{1}{b\sqrt{b^2 - c^2}} \arctan \frac{\sqrt{b^2 - c^2}x}{b} + C. \end{aligned}$$

Let $b = \sqrt{3}$ and $c = \sqrt{2}$, and hence

$$I = \frac{1}{\sqrt{3}\sqrt{3-2}} \arctan \frac{\sqrt{3-2}x}{\sqrt{3}} + C = \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} + C.$$

When $y = 1$, $\frac{\sqrt{2}x}{\sqrt{x^2+1}} = 1$, and hence $x^2 + 1 = 2x^2$, $x^2 = 1$, giving $x = 1$.

When $y \to \sqrt{2} = b$, $x \to \infty$.

Hence,

$$\int_1^{\sqrt{2}} \frac{\mathrm{d}y}{(3 - y^2)\sqrt{2 - y^2}} = \frac{1}{\sqrt{3}} \left[\arctan \frac{x}{\sqrt{3}}\right]_1^\infty = \frac{1}{\sqrt{3}} \left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}.$$

Consider letting $x = \frac{1}{y}$ in the integral, and we have $\mathrm{d}x = -\frac{1}{y^2}\,\mathrm{d}y = -x^2\,\mathrm{d}y$, and when $y = 1$, $x = 1$, and when $y = \sqrt{2}$, $x = \frac{1}{\sqrt{2}}$. Hence,

$$\begin{aligned} \int_{\frac{1}{\sqrt{2}}}^1 \frac{y\,\mathrm{d}y}{(3y^2 - 1)\sqrt{2y^2 - 1}} &= \int_{\sqrt{2}}^1 \frac{\frac{1}{x} \cdot \frac{1}{-x^2}\,\mathrm{d}x}{\left(\frac{3}{x^2} - 1\right)\sqrt{\frac{2}{x^2} - 1}} \\ &= \int_1^{\sqrt{2}} \frac{\mathrm{d}x}{(3 - x^2)\sqrt{2 - x^2}} \\ &= \frac{\pi}{3\sqrt{3}}. \end{aligned}$$

3. Consider the same substitution $y = \frac{ax}{\sqrt{x^2+p}}$. We still have

$$\mathrm{d}y = \frac{ap}{\left(\sqrt{x^2 + p}\right)^3}\,\mathrm{d}x,$$

and hence

$$\begin{aligned} &\int \frac{\mathrm{d}y}{(3y^2 - 1)\sqrt{2y^2 - 1}} \\ &= \int \frac{ap}{\left(\sqrt{x^2 + p}\right)^3} \cdot \frac{\mathrm{d}x}{\left(3 \cdot \frac{a^2x^2}{x^2+p} - 1\right)\sqrt{2 \cdot \frac{a^2x^2}{x^2+p} - 1}} \\ &= \int \frac{ap\,\mathrm{d}x}{(3a^2x^2 - (x^2 + p))\sqrt{2a^2x^2 - (x^2 + p)}} \\ &= \int \frac{ap\,\mathrm{d}x}{((3a^2 - 1)x^2 - p)\sqrt{(2a^2 - 1)x^2 - p}}. \end{aligned}$$

Consider letting $a = \frac{1}{\sqrt{2}}$ and $p = -1$, and we have

$$\begin{aligned}
&\int \frac{\mathrm{d}y}{(3y^2-1)\sqrt{2y^2-1}} \\
={}& \int \frac{-\,\mathrm{d}x}{\sqrt{2}\left(\frac{1}{2}x^2+1\right)} \\
={}& \int \frac{-\sqrt{2}\,\mathrm{d}x}{x^2+2} \\
={}& -\sqrt{2}\cdot\frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}} + C \\
={}& -\arctan\frac{x}{\sqrt{2}} + C.
\end{aligned}$$

When $y = \frac{1}{\sqrt{2}}$, we have $\frac{1}{\sqrt{2}} \cdot \frac{x}{\sqrt{x^2-1}} = \frac{1}{\sqrt{2}}$, and $x \to \infty$. When $y = 1$, we have $\frac{1}{\sqrt{2}} \cdot \frac{x}{\sqrt{x^2-1}} = 1$, and $x = \sqrt{2}$. Hence,

$$\begin{aligned}
&\int_{\frac{1}{\sqrt{2}}}^{1} \frac{\mathrm{d}y}{(3y^2-1)\sqrt{2y^2-1}} \\
={}& -\left[\arctan\frac{x}{\sqrt{2}}\right]_{\infty}^{\sqrt{2}} \\
={}& \left[\arctan\frac{x}{\sqrt{2}}\right]_{\sqrt{2}}^{\infty} \\
={}& \frac{\pi}{2} - \frac{\pi}{4} \\
={}& \frac{\pi}{4}.
\end{aligned}$$

2019.3 Question 6

Notice that the original equation

$$zz^* - az^* - a^*z + aa^* - r^2 = 0$$

can be simplified to

$$(z-a)(z^*-a^*) = r^2,$$

and the left-hand side satisfies

$$(z-a)(z^*-a^*) = (z-a)(z-a)^* = |z-a|^2,$$

which means the original equation is

$$|z-a|^2 = r^2,$$

and hence

$$|z-a| = r.$$

This is a circle centred at a with radius r.

1. Since $\omega = \frac{1}{z}$, we have $z = \frac{1}{\omega}$. Hence,

$$\begin{aligned}
\frac{1}{\omega}\frac{1}{\omega^*} - a\frac{1}{\omega^*} - \frac{1}{\omega}a^* + aa^* &= r^2 \\
1 - \omega a - \omega^* a^* + aa^*\omega\omega^* &= r^2\omega\omega^* \\
(r^2 - aa^*)\omega\omega^* + \omega a + \omega^* a^* &= 1 \\
\omega\omega^* + \frac{a}{r^2-aa^*}\omega + \frac{a^*}{r^2-aa^*}\omega^* &= \frac{1}{r^2-aa^*} \\
\left(\omega + \frac{a^*}{r^2-aa^*}\right)\left(\omega + \frac{a^*}{r^2-aa^*}\right)^* &= \frac{1}{r^2-aa^*} + \frac{aa*}{(r^2-aa^*)^2} \\
\left|\omega - \frac{a^*}{aa^*-r^2}\right|^2 &= \frac{r^2}{(r^2-aa^*)^2} \\
\left|\omega - \frac{a^*}{aa^*-r^2}\right| &= \frac{r}{|r^2-aa^*|},
\end{aligned}$$

so ω is on a circle C' with centre $\frac{a^*}{aa^*-r^2}$ and radius $\frac{r}{|r^2-aa^*|}$.

If C and C' are the same circle, we have

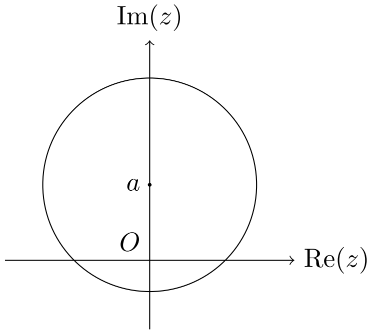
$$a = \frac{a^*}{aa^*-r^2}, r = \frac{r}{|r^2-aa^*|}.$$

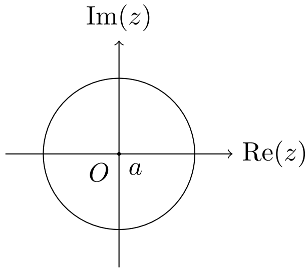
The second equation gives $|r^2-aa^*| = 1$, which means $r^2 - aa^* = \pm 1$.

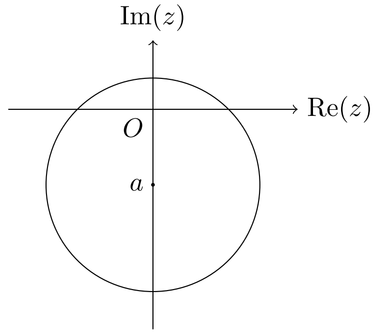
$$\begin{aligned}
r^2 - aa^* &= \pm 1 \\
r^2 - |a|^2 &= \pm 1 \\
\left(|a|^2 - r^2\right)^2 &= 1,
\end{aligned}$$

as desired.

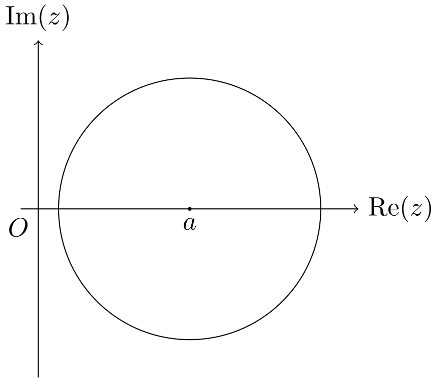
When $r^2 - aa^* = 1$, $a = -a^*$, and hence a is pure imaginary. Since $r^2 = 1 + |a|^2$ in this case, $r > |a|$, so the circle must contain the origin. The diagrams are as below, with the case $\mathrm{Im}(a) > 0$ on the left, $\mathrm{Im}(a) = 0$ in the middle, and $\mathrm{Im}(a) < 0$ on the right:

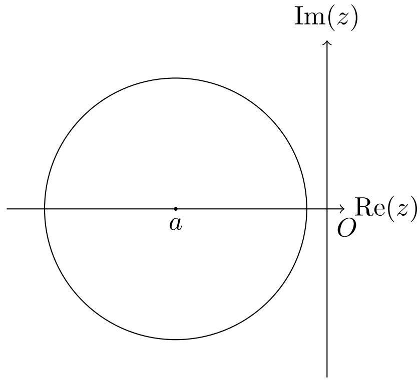






When $r^2 - aa^* = -1$, $a = a^*$, and hence a is real. Since $r^2 = -1 + |a|^2$ in this case, $r < |a|$, so the circle cannot contain the origin, and $|a| > 1$. The diagrams are as below, with the case $\text{Re}(a) > 1$ on the left, and $\text{Re}(a) < -1$ on the right:





2. In the case where $\omega = \frac{1}{z^*}$, we have $z = \frac{1}{\omega^*}$, and hence similar to the previous one,

$$\begin{aligned}
\omega\omega^* + \frac{a}{r^2 - aa^*}\omega^* + \frac{a^*}{r^2 - aa^*}\omega &= \frac{1}{r^2 - aa^*} \\
\left(\omega + \frac{a}{r^2 - aa^*}\right)\left(\omega + \frac{a}{r^2 - aa^*}\right)^* &= \frac{1}{r^2 - aa^*} + \frac{aa*}{(r^2 - aa^*)^2} \\
\left|\omega - \frac{a}{aa^* - r^2}\right|^2 &= \frac{r^2}{(r^2 - aa^*)^2} \\
\left|\omega - \frac{a}{aa^* - r^2}\right| &= \frac{r}{|r^2 - aa^*|},
\end{aligned}$$

so ω is on a circle C' with centre $\frac{a}{aa^* - r^2}$ and radius $\frac{r}{|r^2 - aa^*|}$.

If they are the same circle, we have

$$a = \frac{a}{aa^* - r^2}, r = \frac{r}{|r^2 - aa^*|}.$$

We still have $r^2 - aa^* = \pm 1$.

When $r^2 - aa^* = 1$, we have $a = -a$, and $a = 0$.

When $r^2 - aa^* = -1$, we have $a = a$, and a can be any complex number satisfying $|a| = \sqrt{r^2 + 1}$.

It is not the case that a is either real or pure imaginary.

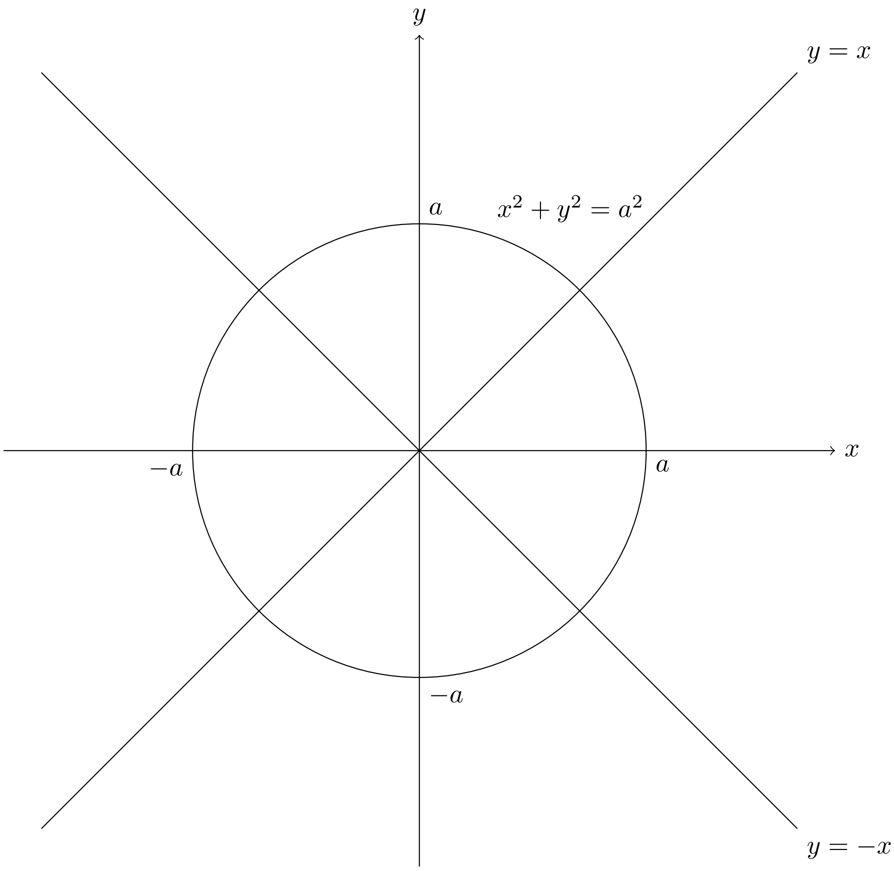
2019.3 Question 7

1. When $a = b$,

$$\begin{aligned} y^2(y^2 - a^2) &= x^2(x^2 - a^2) \\ x^4 - y^4 - a^2x^2 + a^2y^2 &= 0 \\ (x^2 + y^2 - a^2)(x^2 - y^2) &= 0 \\ (x^2 + y^2 - a^2)(x + y)(x - y) &= 0, \end{aligned}$$

so the Devil's Curve in this case consists of the line $x + y = 0$, the line $x - y = 0$, and the circle $x^2 + y^2 = a^2$.

The curve is shown as follows.



2. When $a = 2$ and $b = \sqrt{5}$,

$$y^2(y^2 - 5) = x^2(x^2 - 4).$$

(a) Rearrangement gives us

$$(x^2)^2 - 4x^2 - y^2(y^2 - 5) = 0,$$

and considering the discriminant, we have

$$(-4)^2 + 4y^2(y^2 - 5) \geq 0,$$

i.e.

$$\left(y^2 - 1\right)\left(y^2 - 4\right) \geq 0.$$

This gives $y^2 \leq 1$ or $y^2 \geq 4$, and in the case where $y \geq 0$, this must give $0 \leq y \leq 1$ or $y \geq 2$, as desired.

(b) When the curve is very close to the origin, we must have $x^4, y^4 \ll x^2, y^2$, and hence $4x^2 \approx 5y^2$, which means $y \approx \frac{2}{\sqrt{5}}x$.

When the curve is very far from the origin, we must have $x^4, y^4 \gg x^2, y^2$, and hence $x^4 \approx y^4$, which means $y \approx x$.

(c) Using implicit differentiation, we have

$$\begin{aligned} y^2(y^2-5) &= x^2(x^2-4) \\ (4y^3-10y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 4x^3-8x \\ (2y^2-5)y\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x(x^2-2). \end{aligned}$$

When $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, the tangent to the curve is parallel to the x-axis, and hence

$$2x(x^2-2)=0,$$

giving $x=0$ or $x=\sqrt{2}$.
For $x=0$, $y^2(y^2-5)=0$, and therefore $y=0$ or $y=\sqrt{5}$. The case where $y=0$ does not necessarily give that $\frac{\mathrm{d}y}{\mathrm{d}x}=0$, but the case where $y=\sqrt{5}$ does.
For $x=\sqrt{2}$, $y^2(y^2-5)=-4$, $y=2$ or $y=1$. Both cases give $\frac{\mathrm{d}y}{\mathrm{d}x}=0$.
So the tangent to the curve is parallel to the x-axis at points

$$\left(0,\sqrt{5}\right), \left(\sqrt{2},1\right), \left(\sqrt{2},2\right).$$

We must have

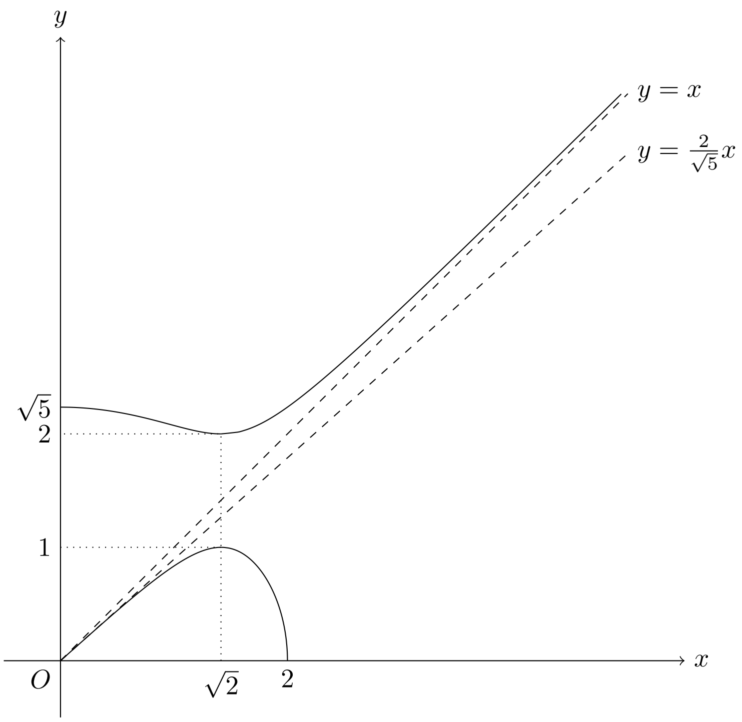
$$(2y^2-5)y=2x(x^2-2)\frac{\mathrm{d}x}{\mathrm{d}y},$$

and when $\frac{\mathrm{d}x}{\mathrm{d}y}=0$, the tangent to the curve is parallel to the y-axis.
This gives $(2y^2-5)y=0$, and hence $y=0$ or $y=\sqrt{\frac{5}{2}}$.
For $y=0$, $x=0$ or $x=2$. The case $x=0$ does not necessarily give $\frac{\mathrm{d}x}{\mathrm{d}y}=0$, but the case where $x=2$ does.
For $y=\sqrt{\frac{5}{2}}$, $x^2(x^2-4)=-\frac{25}{4}$, and hence

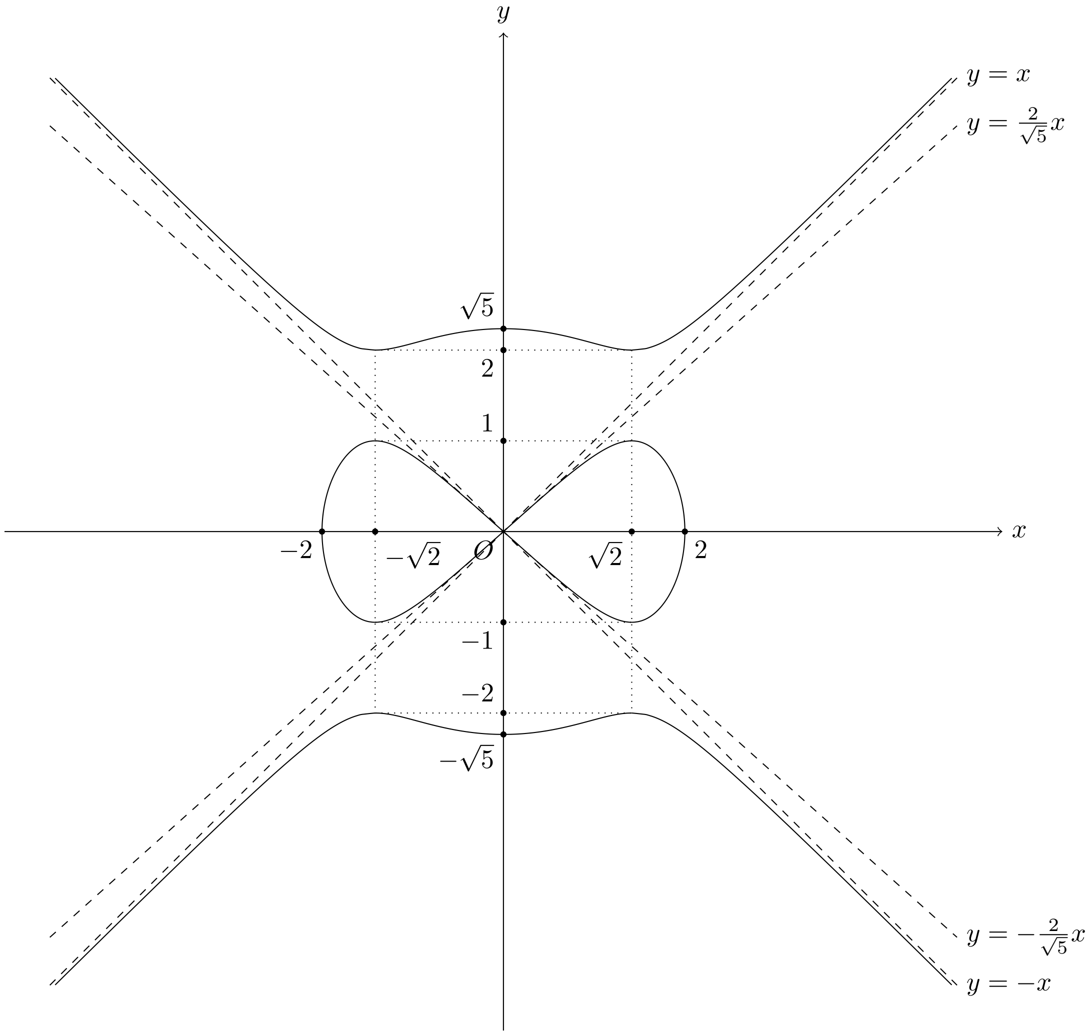
$$4x^4-16x^2+25=4(x^2-2)^2+9=0,$$

which is not possible.
Hence, the tangent to the curve is parallel to the y-axis only at $(2,0)$.

Therefore, from the analysis in the previous parts, the curve looks as follows for $x \geq 0$ and $y \geq 0$:



3. All x terms in the curve is in x^2, so the graph is symmetric in the y-axis since $x^2 = (-x)^2$. Similarly, the graph is symmetric in the x-axis as well. Hence, the complete graph looks as follows.



2019.3 Question 8

1. W.L.O.G. let the origin be the centre of the rectangle $ABCD$ (and let $ABCD$ lie on the x-y plane). We adjust the scale of the axis, and we let $V(0,0,1)$ and $A(-\mu,-\nu,0)$, we have $B(\mu,-\nu,0)$, $C(\mu,\nu,0)$ and $D(-\mu,\nu,0)$. Let $\mu,\nu>0$.

   Let M be the midpoint of AB and N be the midpoint of BC. We must have $M(0,-\nu,0)$ and $N(\mu,0,0)$.

   The angle between the face AVB and the base $ABCD$ must be the angle between $\overrightarrow{MO}$ and $\overrightarrow{MV}$. Hence,

$$\cos\alpha=\frac{\overrightarrow{MO}\cdot\overrightarrow{MV}}{\left|\overrightarrow{MO}\right|\left|\overrightarrow{MV}\right|}.$$

   Note that

$$\overrightarrow{MO}=\begin{pmatrix}0\\\nu\\0\end{pmatrix},\overrightarrow{MV}=\mathbf{v}-\mathbf{m}=\begin{pmatrix}0\\\nu\\1\end{pmatrix},$$

   and hence

$$\cos\alpha=\frac{\nu^2}{\nu\cdot\sqrt{\nu^2+1}}=\frac{\nu}{\sqrt{\nu^2+1}},$$

   which gives

$$\cos^2\alpha\nu^2+\cos^2\alpha=\nu^2,$$

   and hence

$$\sin^2\alpha\nu^2=\cos^2\alpha,$$

   which gives

$$\nu=\cot\alpha.$$

   Similarly,

$$\mu=\cot\beta.$$

   A vector perpendicular to AVB can be

$$\begin{aligned}\overrightarrow{VA}\times\overrightarrow{VB}&=\begin{pmatrix}-\mu\\-\nu\\-1\end{pmatrix}\times\begin{pmatrix}\mu\\-\nu\\-1\end{pmatrix}\\&=\begin{vmatrix}\hat{\mathbf{i}}&\hat{\mathbf{j}}&\hat{\mathbf{k}}\\-\mu&-\nu&-1\\\mu&-\nu&-1\end{vmatrix}\\&=\begin{pmatrix}0\\-2\mu\\2\mu\nu\end{pmatrix}\\&=\begin{pmatrix}0\\-2\cot\beta\\2\cot\alpha\cot\beta.\end{pmatrix}\\&=-\frac{2\cot\beta}{\sin\alpha}\begin{pmatrix}0\\-\sin\alpha\\\cos\alpha\end{pmatrix},\end{aligned}$$

   and so

$$\begin{pmatrix}0\\-\sin\alpha\\\cos\alpha\end{pmatrix}$$

   is a unit vector perpendicular to AVB.

Similarly,

$$\begin{aligned}\overrightarrow{VB} \times \overrightarrow{VC} &= \begin{pmatrix}\mu \\ -\nu \\ -1\end{pmatrix} \times \begin{pmatrix}\mu \\ \nu \\ -1\end{pmatrix} \\ &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \mu & -\nu & -1 \\ \mu & \nu & -1\end{vmatrix} \\ &= \begin{pmatrix}2\nu \\ 0 \\ 2\mu\nu\end{pmatrix} \\ &= \begin{pmatrix}2\cot\alpha \\ 0 \\ 2\cot\alpha\cot\beta\end{pmatrix} \\ &= \frac{2\cot\alpha}{\sin\beta}\begin{pmatrix}\sin\beta \\ 0 \\ \cos\beta\end{pmatrix},\end{aligned}$$

and hence

$$\begin{pmatrix}\sin\beta \\ 0 \\ \cos\beta\end{pmatrix}$$

is a unit vector perpendicular to BVC.

The acute angle between AVB and BVC, θ, satisfies that

$$\cos\theta = \begin{pmatrix}0 \\ -\sin\alpha \\ \cos\alpha\end{pmatrix} \cdot \begin{pmatrix}\sin\beta \\ 0 \\ \cos\beta\end{pmatrix} = \cos\alpha\cos\beta,$$

as desired.

2. Notice that

$$\begin{aligned}\cos\varphi &= \frac{\overrightarrow{BV} \cdot \overrightarrow{BO}}{\left|\overrightarrow{BV}\right| \cdot \left|\overrightarrow{BO}\right|} \\ &= \frac{\begin{pmatrix}-\mu \\ \nu \\ 1\end{pmatrix} \cdot \begin{pmatrix}-\mu \\ \nu \\ 0\end{pmatrix}}{\sqrt{\mu^2+\nu^2+1}\sqrt{\mu^2+\nu^2}} \\ &= \sqrt{\frac{\mu^2+\nu^2}{\mu^2+\nu^2+1}},\end{aligned}$$

and hence

$$\sin\varphi = \sqrt{1-\cos^2\varphi} = \sqrt{\frac{1}{\mu^2+\nu^2+1}},$$

which means

$$\cot\varphi = \sqrt{\mu^2+\nu^2},$$

and hence

$$\cot^2\varphi = \mu^2+\nu^2 = \cot^2\alpha + \cot^2\beta,$$

as desired.

Notice that

$$\begin{aligned}\cos^2\varphi &= \frac{\mu^2+\nu^2}{\mu^2+\nu^2+1} \\ &= \frac{\cot^2\alpha+\cot^2\beta}{\cot^2\alpha+\cot^2\beta+1} \\ &= \frac{\cos^2\alpha\sin^2\beta+\cos^2\beta\sin^2\alpha}{\cos^2\alpha\sin^2\beta+\cos^2\beta\sin^2\alpha+\sin^2\beta\sin^2\alpha} \\ &= \frac{\cos^2\alpha(1-\cos^2\beta)+\cos^2\beta(1-\cos^2\alpha)}{(\cos^2\alpha+\sin^2\alpha)(\cos^2\beta+\sin^2\beta)-\cos^2\alpha\cos^2\beta} \\ &= \frac{\cos^2\alpha+\cos^2\beta-2\cos^2\alpha\cos^2\beta}{1-\cos^2\alpha\cos^2\beta} \\ &= \frac{\cos^2\alpha+\cos^2\beta-2\cos^2\theta}{1-\cos^2\theta}.\end{aligned}$$

Since $(\cos\alpha-\cos\beta)^2=\cos^2\alpha+\cos^2\beta-2\cos\theta\geq 0$, we have $\cos^2\alpha+\cos^2\beta\geq 2\cos\theta$, and hence

$$\cos^2\varphi=\frac{\cos^2\alpha+\cos^2\beta-2\cos^2\theta}{1-\cos^2\theta}\geq\frac{2\cos\theta-2\cos^2\theta}{1-\cos^2\theta}.$$

Notice that

$$\begin{aligned}\cos^2\varphi &\geq \frac{2\cos\theta-2\cos^2\theta}{1-\cos^2\theta} \\ &= \frac{2\cos\theta(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)} \\ &= \frac{2\cos\theta}{1+\cos\theta} \\ &= \frac{2}{1+\cos\theta}\cos\theta \\ &> \frac{2}{1+1}\cos\theta \\ &= \cos\theta \\ &> \cos^2\theta,\end{aligned}$$

since θ is acute, $0<\cos\theta<1$.

This means $\cos^2\varphi>\cos^2\theta$, and since θ,φ are acute, this must mean that $\varphi<\theta$, since $\cos\varphi,\cos\theta$ are both positive, and $\cos\varphi>\cos\theta$.

2019.3 Question 11

1. Let X be the number of customers arriving at builders' merchants on a day, and we have $X \sim \mathrm{Po}(\lambda)$. This means
$$\mathrm{P}(X = x) = \frac{\lambda^x}{e^\lambda x!}$$
for $x = 0, 1, \ldots$.
Let Y be the number of customers taking the sand on a day. Then we have $(Y \mid X = x) \sim \mathrm{B}(x, p)$, and hence
$$\mathrm{P}(Y = y \mid X = x) = \binom{x}{y} p^y (1-p)^{x-y}.$$
Hence, we have
$$\begin{aligned}
\mathrm{P}(Y = y) &= \sum_{x=0}^{\infty} \mathrm{P}(Y = y, X = x) \\
&= \sum_{x=0}^{\infty} \mathrm{P}(Y = y \mid X = x) \, \mathrm{P}(X = x) \\
&= \sum_{x=y}^{\infty} \mathrm{P}(Y = y \mid X = x) \, \mathrm{P}(X = x) \\
&= \sum_{x=y}^{\infty} \binom{x}{y} p^y (1-p)^{x-y} \cdot \frac{\lambda^x}{e^\lambda x!} \\
&= \sum_{x=y}^{\infty} \frac{x! p^y (1-p)^x \lambda^x}{y!(x-y)!(1-p)^y e^\lambda x!} \\
&= \frac{p^y}{y!(1-p)^y e^\lambda} \sum_{x=y}^{\infty} \frac{(1-p)^x \lambda^x}{(x-y)!} \\
&= \frac{p^y}{y!(1-p)^y e^\lambda} \sum_{x=0}^{\infty} \frac{[\lambda(1-p)]^{x+y}}{x!} \\
&= \frac{p^y \lambda^y}{y! e^\lambda} \sum_{x=0}^{\infty} \frac{[\lambda(1-p)]^x}{x!} \\
&= \frac{(p\lambda)^y}{y! e^\lambda} e^{\lambda(1-p)} \\
&= \frac{(p\lambda)^y}{y! e^{p\lambda}},
\end{aligned}$$
which is precisely the probability mass function of $\mathrm{Po}(p\lambda)$, as desired.

2. Let Z be the amount of sand remaining at the end of a day, and hence
$$Z = S(1-k)^Y.$$
Hence, the expectation of Z is given by
$$\begin{aligned}
\mathrm{E}(Z) &= S \, \mathrm{E}\left[(1-k)^Y\right] \\
&= S \sum_{y=0}^{\infty} (1-k)^y \, \mathrm{P}(Y = y) \\
&= \frac{S}{e^{p\lambda}} \sum_{y=0}^{\infty} \frac{(p\lambda(1-k))^y}{y!} \\
&= \frac{S}{e^{p\lambda}} e^{p\lambda(1-k)} \\
&= \frac{S}{e^{pk\lambda}}.
\end{aligned}$$

Let Z' be the amount of sand taken, and hence

$$Z' = S - Z,$$

which means

$$\mathrm{E}(Z') = S - \mathrm{E}(Z) = S\left(1 - e^{-pk\lambda}\right),$$

precisely as desired.

3. Given that $Z = z$, the assistant will take kz of the remaining sand, and the probability of the assistant taking the golden grain event (denoted as G) is

$$\mathrm{P}(G \mid Z = z) = \frac{kz}{S}.$$

Using $Z = S(1-k)^Y$, we have

$$\mathrm{P}(G \mid Y = y) = k(1-k)^y$$

$$\begin{aligned}
\mathrm{P}(G) &= \sum_{y=0}^{\infty} \mathrm{P}(G, Y = y) \\
&= \sum_{y=0}^{\infty} \mathrm{P}(G \mid Y = y)\,\mathrm{P}(Y = y) \\
&= \sum_{y=0}^{\infty} k(1-k)^y \cdot \frac{(p\lambda)^y}{y! e^{p\lambda}} \\
&= \frac{k}{e^{p\lambda}} \sum_{y=0}^{\infty} \frac{(p\lambda(1-k))^y}{y!} \\
&= \frac{k}{e^{p\lambda}} e^{p\lambda(1-k)} \\
&= \frac{k}{e^{pk\lambda}}.
\end{aligned}$$

In the case where $k = 0$, no sand is taken, and hence the probability is 0.

In the case where $k \to 1$, $\mathrm{P}(G) = e^{-p\lambda}$, which is the probability that $Y = 0$. This is precisely when no customer takes any sand (since if any took the sand they must have taken the gold grain), and as $k \to 1$ the merchants' assistant is guaranteed to take the gold provided it is still existent in the final pile.

In the case where $p\lambda > 1$, we differentiate the probability with respect to k, which gives

$$\frac{\mathrm{d}ke^{-pk\lambda}}{\mathrm{d}k} = (1 - pk\lambda)e^{-pk\lambda}.$$

$e^{-pk\lambda}$ is always positive. In the case where $k < \frac{1}{p\lambda}$, $1 - pk\lambda > 0$, and when $k > \frac{1}{p\lambda}$, $1 - pk\lambda < 0$. Hence, precisely when $k = \frac{1}{p\lambda}$, we will have $\mathrm{P}(G)$ taking a maximum, and since $p\lambda > 1$, this k will satisfy $0 < k < 1$ which is within the range.

Hence, the value of k that maximises $\mathrm{P}(G)$ is

$$k = \frac{1}{p\lambda}.$$

2019.3 Question 12

For each integer between 1 to n inclusive, they are either in a subset of S, an element of T, or not. For each integer there are 2 choices, and there are n integers, this means that

$$|T| = 2^n,$$

as desired.

1. Since there is an equal number of sets $B \in T$ for $1 \in B$ and $1 \notin B$, this means

$$\mathrm{P}(1 \in A_1) = \frac{1}{2}.$$

2. For each of the integer $1 \leq t \leq n$, $t \notin A_1 \cap A_2$ if and only if they cannot be in both of A_1 and A_2, and hence

$$\mathrm{P}(t \notin A_1 \cap A_2) = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4},$$

and $A_1 \cap A_2 = \varnothing$ if and only if for all $1 \leq t \leq n$, that $t \notin A_1 \cap A_2$. All these events are independent, and hence

$$\mathrm{P}(A_1 \cap A_2 = \varnothing) = \left(\frac{3}{4}\right)^n.$$

By similar reasoning,

$$\mathrm{P}(A_1 \cap A_2 \cap A_3 = \varnothing) = \left(\frac{7}{8}\right)^n,$$

and

$$\mathrm{P}(A_1 \cap A_2 \cap \cdots \cap A_m = \varnothing) = \left[1 - \left(\frac{1}{2}\right)^m\right]^n = \left(1 - \frac{1}{2^m}\right)^n.$$

3. $A_1 \subseteq A_2$ if and only if for any $1 \leq t \leq n$, we have $t \in A_1 \implies t \in A_2$. For this to happen, either $t \notin A_1$ (in which case we do not worry about whether t is in A_2 or not), or $t \in A_1$ and $t \in A_2$. This means

$$\mathrm{P}(t \in A_1 \implies t \in A_2) = \frac{3}{4},$$

and hence

$$\mathrm{P}(A_1 \subseteq A_2) = \left(\frac{3}{4}\right)^n.$$

For any $1 \leq t \leq n$, $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m$ means we have $t \in A_1 \implies t \in A_2 \implies \cdots \implies t \in A_m$. This happens if and only if $t \in A_i$ gives $t \in A_j$ for all $j \geq i$, and this is true if and only if there exists some $0 \leq k \leq m$, such that for $1 \leq i \leq k$, $t \notin A_k$, and for $k < j \leq m$, $t \in A_k$.

There are precisely $m + 1$ choices for such k, and this means

$$\mathrm{P}(t \in A_1 \implies t \in A_2 \implies \cdots \implies t \in A_m) = \frac{m+1}{2^m},$$

and hence

$$\mathrm{P}(A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m) = \left(\frac{m+1}{2^m}\right)^n,$$

which gives

$$\mathrm{P}(A_1 \subseteq A_2 \subseteq A_3) = \left(\frac{1}{2}\right)^n.$$