Year 2018

2018 Paper 2

2018.2 Question 1

First, notice that $x = 0$ must not be a root to this quartic equation. Therefore, we can divide both sides by x^2, and the original equation is equivalent to

$$x^2 + \frac{1}{x^2} + a\left(x + \frac{1}{x}\right) + b = 0,$$

and this rearranges to

$$\left(x + \frac{1}{x}\right)^2 + a\left(x + \frac{1}{x}\right) + (b - 2) = 0.$$

Notice that

$$k + \frac{1}{k} = \frac{1}{k^{-1}} + k^{-1} = k^{-1} + \frac{1}{k^{-1}},$$

so if $x = k$ satisfies this equation, then $x = k^{-1}$ also satisfies this equation.

Notice that the range of $t = x + \frac{1}{x}$ for non-zero real x is $t \in (-\infty, -2] \cup [2, \infty)$.

Since it is given that all the roots are real, it must be the case that the quadratic equation

$$t^2 + at + (b - 2) = 0$$

produces two real roots situated within $(-\infty, -2] \cup [2, \infty)$.

Notice that for $t \in (-\infty, -2] \cup [2, \infty)$, the equation

$$x + \frac{1}{x} = t$$

has precisely two real roots for $t \neq \pm 2$, and precisely one $x = \pm 1$ for $t = \pm 2$.

1. In this case, by the previous analysis, the only possibility is that $x_1 = x_2 = x_3 = x_4 = \mp 1$. This means that
$$x^4 + ax^3 + bx^2 + ax + 1 = (x \pm 1)^4 = x^4 \pm 4x^3 + 6x^2 \pm 4x + 1,$$
and hence $(a, b) = (\pm 4, 6)$.

2. Since there are exactly three distinct roots for x, this means that the one which repeated must be $x_1 = x_2 = \pm 1$, which leads to $t_1 = \pm 2$, and those two which does not leads to $t_2 \neq \pm 2$.

   Putting $t_1 = \pm 2$ into the quadratic equation in t, we have
$$4 \pm 2a + (b - 2) = 0,$$
and hence
$$b = \mp 2a - 2,$$
precisely as desired.

3. When $b = 2a - 2$, we have
$$t^2 + at + (2a - 4) = 0,$$
which solves to $t_1 = -2$, $t_2 = -a + 2$.

   For $x + \frac{1}{x} = t_1 = -2$, this solves to $x_1 = x_2 = -1$.

   For $x + \frac{1}{x} = t_2 = -a + 2$, this rearranges to
$$x^2 + (a - 2)x + 1 = 0,$$
and hence the two roots are
$$x_{3,4} = \frac{-(a-2) \pm \sqrt{(a-2)^2 - 4}}{2} = \frac{-a + 2 \pm \sqrt{a^2 - 4a}}{2}$$

4. We first look at necessary condition. Given the equation has precisely two roots, we have $b = \pm 2a - 2$, and hence the quadratic equation in t becomes

$$t^2 + at + (\pm 2a - 4) = 0.$$

$t_1 = \mp 2$ must be a root, and notice that this factorises to

$$t^2 + at + (\pm 2a - 4) = (t \pm 2)(t - (-a \pm 2)),$$

and hence the other root is $t_2 = -a \pm 2$.

As discussed before, we must have that $t_2 < -2$ or $t_2 > 2$ to produce two distinct roots for x, and hence

$$-a \pm 2 < -2 \text{ or } -a \pm 2 > 2,$$

and hence

$$a \mp 2 > 2 \text{ or } a \mp 2 < -2,$$

and hence

$$a > 2 \pm 2 \text{ or } a < -2 \pm 2.$$

Therefore, a necessary condition is $b = \pm 2a - 2$, and $a \in (-\infty, -2 \pm 2) \cup (2 \pm 2, \infty)$.

We would like to show that this is a sufficient condition as well. If $b = \pm 2a - 2$ and $a \in (-\infty, -2 \pm 2) \cup (2 \pm 2, \infty)$, we have the quadratic in t simplifies to

$$t^2 + at + (\pm 2a - 4) = (t \pm 2)(t - (-a \pm 2)) = 0.$$

This gives roots $t_1 = \mp 2$ which in turn gives $x_1 = x_2 = \mp 1$, and $t_2 = -a \pm 2$. In the second case, since

$$a \in (-\infty, -2 \pm 2) \cup (2 \pm 2, \infty),$$

we must have

$$a \mp 2 \in (-\infty, -2) \cup (2, \infty)$$

and hence

$$-a \pm 2 \in (-\infty, -2) \cup (2, \infty).$$

This shows that there are two distinct xs corresponding to t_2, both of which are not equal to ± 1.

Hence, in this case, the original equation has 3 distinct roots precisely, and

$$b = \pm 2a - 2, a \in (-\infty, -2 \pm 2) \cup (2 \pm 2, \infty)$$

is a necessary and sufficient condition for the original equation to have precisely 3 distinct real roots.

The following is to simplify this to what is written in the mark scheme. $b = \pm 2a - 2$ is equivalent to $b + 2 = \pm 2a$, and $(b + 2)^2 = 4a^2$.

The second part is equivalent to $a \mp 2 \in (-\infty, -2) \cup (2, \infty)$, i.e.

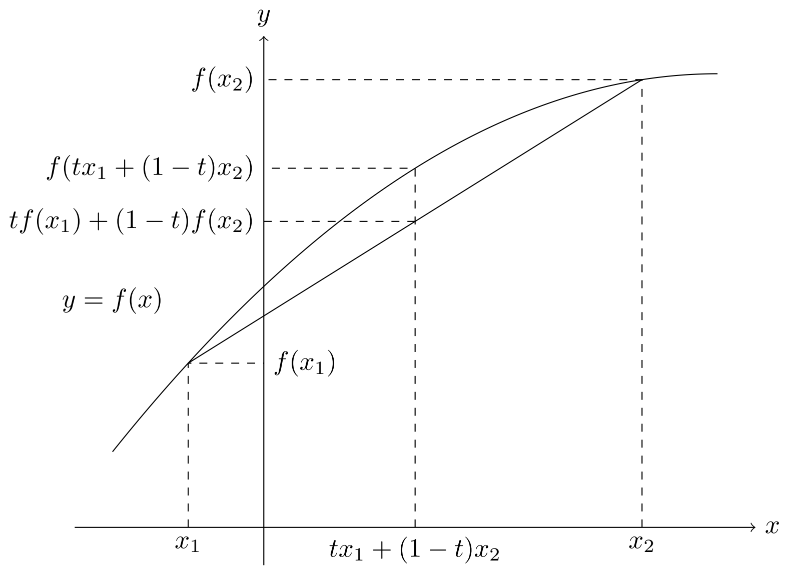
$$(a \mp 2)^2 = a^2 \mp 4a + 4 > 4,$$

i.e.

$$a^2 > \pm 4a = 2 \pm 2a = 2(b + 2) = 2b + 4,$$

precisely what is in the mark scheme.

2018.2 Question 2



If $f''(x) < 0$, this means $f'(x)$ is decreasing, i.e. the gradient of a tangent to the curve $y = f(x)$ is decreasing. Assume, B.W.O.C., that some $f(x)$ satisfies this condition but is not convex. This means that there exists some $a < x_1 < x_2 < b$ and some $0 < t < 1$ that

$$tf(x_1) + (1-t)f(x_2) \geq f(tx_1 + (1-t)x_2).$$

This means that some point on the chord connecting $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is above the graph of the function at that point with x-coordinate $tx_1 + (1-t)x_2$. Hence, the gradient of that function must be less than the gradient of the chord at that point, and since $f''(x) < 0$, the function must continue to have a gradient of less than this, and hence cannot pass through $(x_2, f(x_2))$.

Hence, this triple of (x_1, x_2, t) does not exist, and the function f must be concave on (a, b).

1. Let $x_1 = \frac{2u+v}{3}$ and $x_2 = \frac{v+2w}{3}$, and let $t = \frac{1}{2}$. We can see that $a < x_1, x_2 < b$ and hence we have

$$\frac{1}{2}f(x_1) + \frac{1}{2}f(x_2) \leq f\left(\frac{1}{2}x_1 + \frac{1}{2}x_2\right),$$

which gives

$$\frac{1}{2}f\left(\frac{2u+v}{3}\right) + \frac{1}{2}f\left(\frac{v+2w}{3}\right) \leq f\left(\frac{u+v+w}{3}\right).$$

Let $x_1 = u$ and $x_2 = v$, and let $t = \frac{2}{3}$. We have

$$\frac{2}{3}f(u) + \frac{1}{3}f(v) \leq f\left(\frac{2u+v}{3}\right),$$

and let $x_1 = w$, $x_2 = v$, and let $t = \frac{2}{3}$, we have

$$\frac{2}{3}f(w) + \frac{1}{3}f(v) \leq f\left(\frac{2w+v}{3}\right).$$

Hence,

$$\begin{aligned}
f\left(\frac{u+v+w}{3}\right) &\geq \frac{1}{2}f\left(\frac{2u+v}{3}\right) + \frac{1}{2}f\left(\frac{v+2w}{3}\right) \\
&\geq \frac{1}{2} \cdot \left[\frac{2}{3}f(u) + \frac{1}{3}f(v)\right] + \frac{1}{2} \cdot \left[\frac{2}{3}f(w) + \frac{1}{3}f(v)\right] \\
&= \frac{1}{3}\left[f(u) + f(v) + f(w)\right],
\end{aligned}$$

which shows exactly what is desired.

2. Let $a = 0$ and $b = \pi$, and let $f(x) = \sin x$. We aim to show that f is concave, and notice that
$$f''(x) = -\sin x < 0$$
for all $0 < x < \pi$, so it is concave on $(0, \pi)$.
Angles in a triangle lie within $(0, \pi)$, and they must sum up to π. Hence, by applying the previous part, we have
$$\sin A + \sin B + \sin C \leq 3 \sin\left(\frac{A+B+C}{3}\right) = 3 \sin\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2},$$
as desired.

3. We keep $a = 0$ and $b = \pi$, and let $f(x) = \ln \sin x$. Note that
$$f'(x) = \frac{\cos x}{\sin x} = \cot x,$$
and hence
$$f''(x) = -\csc^2 x < 0$$
which shows that f is concave on $(0, \pi)$.
Hence,
$$\begin{aligned}\ln(\sin A \sin B \sin C) &= \ln \sin A + \ln \sin B + \ln \sin C \\ &\leq 3 \ln \sin\left(\frac{A+B+C}{3}\right) \\ &= 3 \ln \sin\left(\frac{\pi}{3}\right) \\ &= 3 \ln \frac{\sqrt{3}}{2} \\ &= \ln \frac{3\sqrt{3}}{8}.\end{aligned}$$
Since ln is a strictly increasing function, we can then conclude that
$$\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8},$$
as desired.

2018.2 Question 3

1. Notice that

$$\begin{aligned}
f'(x) &= -\frac{\sec^2 x}{(1+\tan x)^2} \\
&= -\frac{1}{\cos^2 x\,(1+\tan x)^2} \\
&= -\frac{1}{(\sin x+\cos x)^2} \\
&= -\frac{1}{\sin^2 x+\cos^2 x+2\sin x\cos x} \\
&= -\frac{1}{1+\sin 2x},
\end{aligned}$$

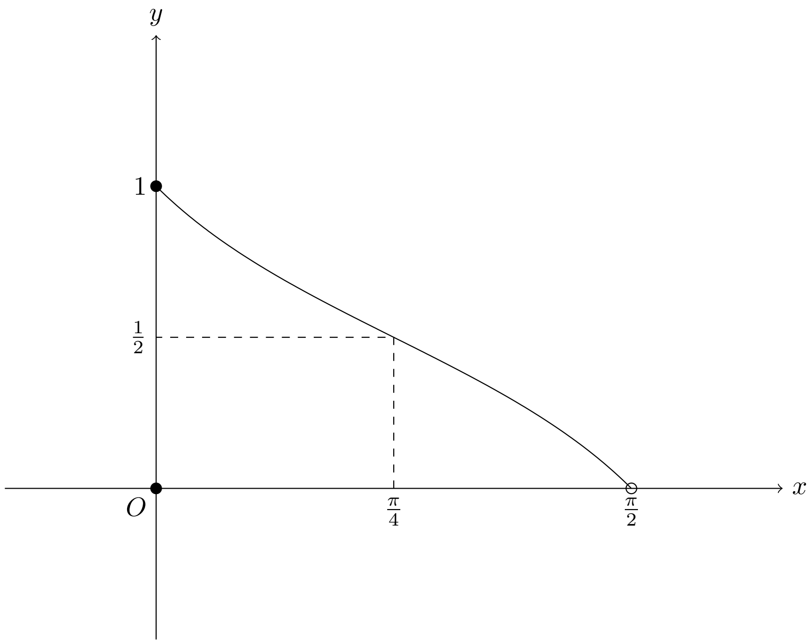
as desired.

Since $0 \leq x < \frac{1}{2}\pi$, $0 \leq 2x < \pi$, and hence $0 \leq \sin 2x \leq 1$.

This means that $-1 \leq f'(x) \leq -\frac{1}{2}$.

$\sin 2x$ increases on $\left(0, \frac{\pi}{4}\right)$ and decreases on $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

Hence, the graph must look as follows.



2. If $y = g(x)$ has rotational symmetry about (a, b), then this means if point $(a + x, b + y)$ is on the graph, then the point $(a - x, b - y)$ is on the graph as well.

This means that $g(a + x) + g(a - x) = (b + y) + (b - y) = 2b$, and setting $x' = a + x$ gives $g(x') + g(2a - x') = 2b$ gives precisely what is desired.

On the other hand, if for all x, $g(x) + g(2a - x) = 2b$, then points $(x, g(x))$ and $(2a - x, g(2a - x))$ on the graph, have midpoint

$$\left(\frac{x + (2a - x)}{2}, \frac{g(x) + g(2a - x)}{2}\right) = (a, b)$$

is the desired centre of symmetry. This means each point on the graph corresponds to another point on the graph when mirrored through the desired centre of symmetry, showing it has rotational symmetry of order 2 about that point, precisely as desired.

The integral evaluates to zero.

3. We would like to show that this function has rotational symmetry about the point $\left(\frac{\pi}{4}, \frac{1}{2}\right)$. Notice that

$$\begin{aligned}
y|_x + y|_{2\cdot\frac{\pi}{4}-x} &= \frac{1}{1+\tan^k x} + \frac{1}{1+\tan^k\left(\frac{\pi}{2}-x\right)} \\
&= \frac{1}{1+\tan^k x} + \frac{1}{1+\cot^k x} \\
&= \frac{1}{1+\tan^k x} + \frac{\tan^k x}{\tan^k x + 1} \\
&= \frac{1+\tan^k x}{1+\tan^k x} \\
&= 1 \\
&= 2 \cdot \frac{1}{2},
\end{aligned}$$

which shows the rotational symmetry.

Hence,

$$\begin{aligned}
\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1+\tan^k x}\,\mathrm{d}x &= \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} y|_x\,\mathrm{d}x + \int_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} y|_x\,\mathrm{d}x \\
&= \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} y|_x\,\mathrm{d}x + \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} y|_{\frac{\pi}{2}-x}\,\mathrm{d}x \\
&= \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \left[y|_x + y|_{2\cdot\frac{\pi}{4}-x}\right]\mathrm{d}x \\
&= \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \mathrm{d}x \\
&= \frac{1}{4}\pi - \frac{1}{6}\pi \\
&= \frac{1}{12}\pi.
\end{aligned}$$

2018.2 Question 4

1. By the identity, we have

$$\cos x + \cos 4x = 2\cos\frac{5}{2}x\cos\frac{3}{2}x,$$

and

$$\cos 2x + \cos 3x = 2\cos\frac{5}{2}x\cos\frac{1}{2}x.$$

Hence, we have

$$\cos x + 3\cos 2x + 3\cos 3x = 2\cos\frac{5}{2}x\left(\cos\frac{3}{2}x + 3\cos\frac{1}{2}x\right) = 0.$$

Hence, either

$$\cos\frac{5}{2}x = 0,$$

or

$$\cos\frac{3}{2}x + 3\cos\frac{1}{2}x = 0.$$

In the first case, we have $\frac{5}{2}x = \frac{1}{2}\pi + k\pi$ for $k \in \mathbb{Z}$, and hence

$$x = \frac{1+2k}{5}\cdot\pi.$$

Since $0 \leq x \leq 2\pi$, we have

$$0 \leq \frac{1+2k}{5} \leq 2,$$

and hence

$$0 \leq 1 + 2k \leq 10,$$

giving $k = 0, 1, 2, 3, 4$. Hence, the solutions are

$$x = \frac{1}{5}\pi, x = \frac{3}{5}\pi, x = \pi, x = \frac{7}{5}\pi, x = \frac{9}{5}\pi.$$

In the second case, notice that

$$\begin{aligned}\cos 3t &= \cos(2t + t)\\ &= \cos 2t\cos t - \sin 2t\sin t\\ &= (\cos^2 t - \sin^2 t)\cos t - 2\sin^2 t\cos t\\ &= \cos^3 t - 3\sin^2\cos t.\end{aligned}$$

Hence,

$$\cos\frac{3}{2}x + 3\cos\frac{1}{2}x = 0 \iff \cos^3\frac{1}{2}x - 3\sin^2\frac{1}{2}x\cos\frac{1}{2}x + 3\cos\frac{1}{2}x = 0,$$

and using the identity $\sin^2 t + \cos^2 t = 1$, this simplifies to

$$\cos^3\frac{1}{2}x + 3\cos^3\frac{1}{2}x = 0,$$

which is

$$\cos\frac{1}{2}x = 0.$$

This gives

$$\frac{1}{2}x = \frac{\pi}{2} + k\pi$$

for $k \in \mathbb{Z}$, and hence

$$x = (1 + 2k)\pi.$$

Since $0 \leq x \leq 2\pi$, the only k valid is $k = 0$, and this solves to $x = \pi$.

Hence, all the solutions to this equation is

$$x \in \left\{\frac{1}{5}\pi, \frac{3}{5}\pi, \pi, \frac{7}{5}\pi, \frac{9}{5}\pi\right\}.$$

2. Using the given identity, we have
$$\cos(x+y) + \cos(x-y) = 2\cos x \cos y.$$
Hence, the original equation simplifies to
$$2\cos x \cos y - \cos 2x = 1.$$
Using the identity $\cos 2x = 2\cos^2 x - 1$, and this gives
$$2\cos x \cos y - (2\cos^2 x - 1) = 1,$$
and hence
$$2\cos x \cos y - 2\cos^2 x = 0,$$
which means
$$\cos x(\cos y - \cos x) = 0,$$
and hence $\cos x = 0$ or $\cos y - \cos x = 0$.
The first one gives us $x = \frac{\pi}{2}$ in the range $x \in [0, \pi]$.
Since cos is one-to-one when restricted to $[0, \pi]$, the second one is equivalent to $\cos y = \cos x$ which is equivalent to $x = y$.
The specific value is $x = \frac{\pi}{2}$.

3. Using the identity given, we have
$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2},$$
and
$$\cos(x+y) = 2\cos^2\frac{x+y}{2} - 1.$$
Let $u = \frac{x+y}{2}$ and $v = \frac{x-y}{2}$. We have $0 \leq u \leq \pi$ and $-\frac{\pi}{2} \leq v \leq \frac{\pi}{2}$, and the original equation simplifies to
$$2\cos u \cos v - 2\cos^2 u + 1 = \frac{3}{2},$$
and hence
$$4\cos u \cos v - 4\cos^2 u + 2 = 3,$$
and
$$4\cos^2 u - 4\cos u \cos v + 1 = 0.$$
Since $1 = \cos^2 v + \sin^2 v$, we have
$$4\cos^2 u - 4\cos u \cos v + \cos^2 v = -\sin^2 v,$$
and hence
$$(2\cos u - \cos v)^2 = -\sin^2 v.$$
The left-hand side is non-negative, and the right-hand side is non-positive. Hence, the only way for the equal sign to take place is when both sides are zero, which is
$$2\cos u = \cos v, \sin v = 0.$$
Within this range of v, the only case where $\sin v = 0$ is when $v = 0$, and hence $2\cos u = 1$, $\cos u = \frac{1}{2}$, leading to $u = \frac{\pi}{3}$.
Hence, $x = u + v = \frac{\pi}{3}$, and $y = u - v = \frac{\pi}{3}$, and the only solution is
$$(x, y) = \left(\frac{\pi}{3}, \frac{\pi}{3}\right).$$

2018.2 Question 5

1. For $|x| < 1$, we have
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{t=0}^{\infty} (-x)^t.$$
Since $\ln(1+x)$ differentiates to $\frac{1}{1+x}$, by integration, we have
$$\begin{aligned}\ln(1+x) &= \int \frac{1}{1+x}\,\mathrm{d}x \\ &= \int \sum_{t=0}^{\infty}(-x)^t\,\mathrm{d}x \\ &= \sum_{t=0}^{\infty}(-1)^t \int x^t\,\mathrm{d}x \\ &= C + \sum_{t=0}^{\infty}(-1)^t \frac{x^{t+1}}{t+1} \\ &= C - \sum_{t=1}^{\infty} \frac{(-x)^t}{t}.\end{aligned}$$
Let $x = 0$, and we see $\ln(1+x) = \ln 1 = 0$, and the sum on the right-hand side evaluates to 0, and hence $C = 0$. This gives the Maclaurin expansion for $\ln(1+x)$
$$\ln(1+x) = -\sum_{t=1}^{\infty} \frac{(-x)^t}{t}.$$

2. We have
$$e^{-ax} = \sum_{t=0}^{\infty} \frac{(-ax)^t}{t!},$$
and hence
$$\begin{aligned}&\int_0^\infty \frac{(1-e^{-ax})e^{-x}}{x}\,\mathrm{d}x \\ =& \int_0^\infty \frac{-\sum_{t=1}^{\infty}\frac{(-ax)^t}{t!}\cdot e^{-x}}{x}\,\mathrm{d}x \\ =& \sum_{t=1}^{\infty}\int_0^\infty \frac{-(-ax)^t e^{-x}}{t!x}\,\mathrm{d}x \\ =& \sum_{t=1}^{\infty}\int_0^\infty \frac{(-x)^{t-1}a^t e^{-x}}{t!}\,\mathrm{d}x \\ =& \sum_{t=1}^{\infty}\frac{(-1)^{t-1}a^t}{t!}\int_0^\infty x^{t-1}e^{-x}\,\mathrm{d}x.\end{aligned}$$
We aim to find an expression for
$$I_t = \int_0^\infty x^t e^{-x}\,\mathrm{d}x.$$

Using integration by parts, we have

$$\begin{aligned}
I_t &= \int_0^\infty x^t e^{-x} \, \mathrm{d}x \\
&= -\int_0^\infty x^t \, \mathrm{d}e^{-x} \\
&= -\left[\left(x^t e^{-x}\right)_0^\infty - \int_0^\infty e^{-x} \, \mathrm{d}x^t \right] \\
&= t \int_0^\infty e^{-x} x^{t-1} \, \mathrm{d}x \\
&= t I_{t-1},
\end{aligned}$$

and further noticing that

$$I_0 = \int_0^\infty e^{-x} \, \mathrm{d}x = \left[-e^{-x}\right]_0^\infty = 1,$$

we can see

$$I_t = t!,$$

and hence

$$\begin{aligned}
&\int_0^\infty \frac{(1-e^{-ax})e^{-x}}{x} \, \mathrm{d}x \\
&= \sum_{t=1}^\infty \frac{(-1)^{t-1} a^t}{t!} \int_0^\infty x^{t-1} e^{-x} \, \mathrm{d}x \\
&= \sum_{t=1}^\infty \frac{(-1)^{t-1} a^t}{t!} (t-1)! \\
&= \sum_{t=1}^\infty \frac{(-1)^{t-1} a^t}{t} \\
&= -\sum_{t=1}^\infty \frac{(-a)^t}{t} \\
&= \ln(1+a),
\end{aligned}$$

precisely as desired.

3. Using a substitution $x = e^{-u}$, when $x = 1$, $u = 0$, and when $x \to 0^+$, $u \to \infty$. Also,

$$\frac{\mathrm{d}x}{\mathrm{d}u} = -e^{-u},$$

and hence

$$\begin{aligned}
&\int_0^1 \frac{x^p - x^q}{\ln x} \, \mathrm{d}x \\
&= \int_\infty^0 \frac{e^{-up} - e^{-uq}}{\ln e^{-u}} \cdot \left(-e^{-u}\right) \mathrm{d}u \\
&= \int_\infty^0 \frac{\left(e^{-up} - e^{-uq}\right) e^{-u}}{u} \, \mathrm{d}u \\
&= \int_0^\infty \frac{\left[\left(1 - e^{-up}\right) + \left(1 - e^{-uq}\right)\right] e^{-u}}{u} \, \mathrm{d}u \\
&= \int_0^\infty \frac{\left(1 - e^{-up}\right) e^{-u}}{u} \, \mathrm{d}u - \int_0^\infty \frac{\left(1 - e^{-uq}\right) e^{-u}}{u} \, \mathrm{d}u \\
&= \ln(1+p) - \ln(1+q).
\end{aligned}$$

2018.2 Question 6

1. Notice that for $n \geq 5$, $n! = 5 \cdot 4! \cdot 6 \cdot 7 \cdots n$, and $n! = 5k$ for $k = 4! \cdot 6 \cdot 7 \cdots n > 1$ is an integer.

   Therefore,
   $$n! + 5 = 5k + 5 = 5(k + 1)$$
   is a multiple of two integers greater than 1, and hence p cannot be prime.

   Hence, $n < 5$.

   If $n = 1$, $n! + 5 = 6$ is not prime.

   If $n = 2$, $n! + 5 = 7$ is prime. $(n, p) = (2, 7)$ is a solution.

   If $n = 3$, $n! + 5 = 11$ is prime. $(n, p) = (3, 11)$ is a solution.

   If $n = 4$, $n! + 5 = 29$ is prime. $(n, p) = (4, 29)$ is a solution.

   Therefore, all solutions are $(n, p) = (2, 7), (3, 11)$ and $(4, 29)$.

2. If $n \geq 7$, then we have
   $$m! = 1! \times 3! \times \cdots \times (2n - 1)! > (4n)!$$
   and hence $m > 4n$.

   Let p be some prime number between $2n$ and $4n$. Therefore, $m!$ must include p as one of its terms, and $p \mid m! = \text{RHS}$.

   However, on the left-hand side, all the terms are less than p, and since p is a prime, it must not divide any term in the left-hand side factorial expansion (since every term in the expansion is less than p), and hence $p \nmid \text{LHS}$.

   But since LHS = RHS this is impossible, and we can deduce that $n < 7$.

   - $n = 1$, $\text{LHS} = 1! = 1$ and $(n, m) = (1, 1)$ is a solution.
   - $n = 2$, $\text{LHS} = 1! \cdot 3! = 3!$ and $(n, m) = (2, 3)$ is a solution.
   - $n = 3$, $\text{LHS} = 1! \cdot 3! \cdot 5! = 6 \cdot 5! = 6!$ and $(n, m) = (3, 6)$ is a solution.
   - $n = 4$, $\text{LHS} = 1! \cdot 3! \cdot 5! \cdot 7! = 6! \cdot 7! = 7! \cdot 6! = 7! \cdot (3 \cdot 2 \cdot 5 \cdot 4 \cdot 3 \cdot 2) = 7! \cdot (2 \cdot 4) \cdot (3 \cdot 3) \cdot (2 \cdot 5) = 10!$ and $(n, m) = (4, 10)$ is a solution.
   - $n = 5$, $\text{LHS} = 1! \cdot 3! \cdot 5! \cdot 7! \cdot 9! = 10! \cdot 9! > 10!$, so if m exists, $m > 10$ and $m \geq 11$. Then $11 \mid \text{RHS} = \text{LHS}$, but this is impossible since $11 > 9$, so such m does not exist.
   - $n = 6$, $\text{LHS} = 1! \cdot 3! \cdot 5! \cdot 7! \cdot 9! \cdot 11! = 10! \cdot 9! \cdot 11! = 11! \cdot 9! \cdot 10! = 12! \cdot 10! \cdot (9 \cdot 8 \cdot 7 \cdot 5 \cdot 4 \cdot 3) > 12!$, so if m exists, $m > 12$ and $m \geq 13$. Then $13 \mid \text{RHS} = \text{LHS}$, but this is impossible since $13 > 11$, and so such m does not exist.

   Hence, the only possible solutions are
   $$(n, m) \in \{(1, 1), (2, 3), (3, 6), (4, 10)\}.$$

2018.2 Question 7

Since $|MQ| = \mu|QB|$, we must have $|MQ| = \frac{\mu}{1+\mu}|MB|$, and hence

$$\overrightarrow{MQ} = \frac{\mu}{1+\mu}\overrightarrow{MB},$$

and hence

$$\mathbf{q} - \mathbf{m} = \frac{\mu}{1+\mu}\left(\mathbf{b} - \mathbf{m}\right).$$

Similarly,

$$\mathbf{q} - \mathbf{n} = \frac{\nu}{1+\nu}\left(\mathbf{a} - \mathbf{n}\right).$$

Since $\mathbf{q} = \mathbf{q}$, we have

$$\frac{\mu}{1+\mu}\left(\mathbf{b} - \mathbf{m}\right) + \mathbf{m} = \frac{\nu}{1+\nu}\left(\mathbf{a} - \mathbf{n}\right) + \mathbf{n},$$

which rearranges to give

$$\frac{1}{1+\mu}\mathbf{m} - \frac{1}{1+\nu}\mathbf{n} = \frac{\nu}{1+\nu}\mathbf{a} - \frac{\mu}{1+\mu}\mathbf{b}.$$

Since $\mathbf{m}$ is a scalar multiple of $\mathbf{a}$ as M is on the side OA, and $\mathbf{n}$ is a scalar multiple of $\mathbf{b}$ similarly, and $\mathbf{a}$ and $\mathbf{b}$ are linearly independent since OAB forms a triangle, we can conclude that

$$\mathbf{m} = \frac{1+\mu}{1} \cdot \frac{\nu}{1+\nu}\mathbf{a},$$

and hence

$$\mathbf{m} = \frac{(1+\mu)\nu}{1+\nu}\mathbf{a}.$$

Similarly,

$$\mathbf{n} = \frac{(1+\nu)\mu}{1+\mu}\mathbf{b}.$$

Since L lies on OB with $|OL| = \lambda|OB|$, then we have

$$\mathbf{l} = \lambda\mathbf{b},$$

and hence

$$\overrightarrow{ML} = \mathbf{l} - \mathbf{m} = \lambda\mathbf{b} - \frac{(1+\mu)\nu}{1+\nu}\mathbf{a}.$$

Since

$$\overrightarrow{AN} = \mathbf{n} - \mathbf{a} = \frac{(1+\nu)\mu}{1+\mu}\mathbf{b} - \mathbf{a}.$$

$\overrightarrow{ML}$ is parallel to $\overrightarrow{AN}$ means that the corresponding scalar vectors for $\mathbf{a}$ and $\mathbf{b}$ are in ratio (since they are linearly independent), and hence

$$\lambda : \frac{(1+\nu)\mu}{1+\mu} = \left(-\frac{(1+\mu)\nu}{1+\nu}\right) : (-1),$$

and hence

$$\lambda = \frac{(1+\mu)\nu}{1+\nu} \cdot \frac{(1+\nu)\mu}{1+\mu} = \mu\nu.$$

The condition $\mu\nu < 1$ ensured that L lies on OB between O and B (i.e. on the side OB).

2018.2 Question 8

1. Since $v = \sqrt{y}$, we have $y = v^2$, and hence
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2v\frac{\mathrm{d}v}{\mathrm{d}t},$$
and hence the original equation reduces to
$$2v\frac{\mathrm{d}v}{\mathrm{d}t} = \alpha v - \beta v^2,$$
which gives
$$2\frac{\mathrm{d}v}{\mathrm{d}t} = \alpha - \beta v.$$
Rearranging gives us
$$\frac{\mathrm{d}v}{\alpha - \beta v} = \frac{\mathrm{d}t}{2},$$
and hence integrating both sides gives
$$-\frac{1}{\beta}\ln|\alpha - \beta v| = \frac{1}{2}t + C.$$
Hence,
$$\ln|\alpha - \beta v| = -\frac{\beta t}{2} + C',$$
and
$$\alpha - \beta v = A\exp\left(-\frac{\beta t}{2}\right),$$
and hence
$$v = \frac{1}{\beta}\left[\alpha + A\exp\left(-\frac{\beta}{2}\right)\right],$$
which means
$$y = v^2 = \frac{1}{\beta^2}\left[\alpha + A\exp\left(-\frac{\beta t}{2}\right)\right]^2.$$
Since $y = 0$ when $t = 0$, we have $A = -\alpha$, and hence
$$y_1(t) = \frac{\alpha^2}{\beta^2}\left[1 - \exp\left(-\frac{\beta t}{2}\right)\right]^2.$$

2. Let $v = \sqrt[3]{y}$ in this case, and hence $y = v^3$, we have
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3v^2\frac{\mathrm{d}v}{\mathrm{d}t},$$
and hence the original equation reduces to
$$3v^2\frac{\mathrm{d}v}{\mathrm{d}t} = \alpha v^2 - \beta v^3,$$
and hence
$$3\frac{\mathrm{d}v}{\mathrm{d}t} = \alpha - \beta v.$$
Similar to before, this solves to
$$v = \frac{1}{\beta}\left[\alpha + B\exp\left(-\frac{\beta}{3}\right)\right],$$
and hence
$$y = v^3 = \frac{1}{\beta^3}\left[\alpha + B\exp\left(-\frac{\beta}{3}\right)\right]^3.$$
Since $y = 0$ when $t = 0$, we have $B = -\alpha$, and hence
$$y_2(t) = \frac{\alpha^3}{\beta^3}\left[1 - \exp\left(-\frac{\beta t}{3}\right)\right]^3.$$

3. Let $\alpha = \beta = \gamma$. We have

$$y_1(t) = \left[1 - \exp\left(-\frac{\gamma t}{2}\right)\right]^2, y_2(t) = \left[1 - \exp\left(-\frac{\gamma t}{3}\right)\right]^3.$$

For $t > 0$, we have

$$0 > -\frac{\gamma t}{3} > -\frac{\gamma t}{2} > -\infty,$$

and since the exponential function is strictly increasing, we have

$$1 > \exp\left(-\frac{\gamma t}{3}\right) > \exp\left(-\frac{\gamma t}{2}\right) > 0,$$

and hence

$$1 > 1 - \exp\left(-\frac{\gamma t}{2}\right) > 1 - \exp\left(-\frac{\gamma t}{3}\right) > 0.$$

Hence,

$$y_1(t) = \left[1 - \exp\left(-\frac{\gamma t}{2}\right)\right]^2 > \left[1 - \exp\left(-\frac{\gamma t}{3}\right)\right]^2 > \left[1 - \exp\left(-\frac{\gamma t}{3}\right)\right]^3 = y_2(t)$$

which tells us that the graph of y_2 should lie below the graph of y_1.

As $t \to \infty$,

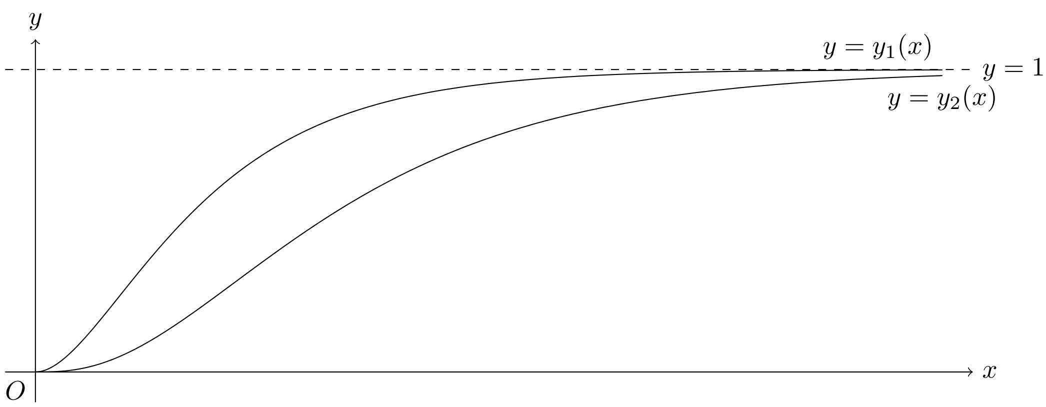
$$\exp\left(-\frac{\gamma t}{2}\right), \exp\left(-\frac{\gamma t}{2}\right) \to 0^+,$$

and hence

$$y_1(t), y_2(t) \to 1^-.$$

At $t = 0$, $y_1(t) = y_2(t) = 0$, and hence by the original differential equation $y_1'(t) = y_2'(t) = 0$.

Hence, the graph looks as follows.



2018.2 Question 12

1. If h consecutive heads are thrown, then the person will earn £h, and the probability of this happening is p^h.

   If this did not happen, then the game must have already ended before reaching h heads (since there must be a tail), and the person will earn nothing.

   Hence, the expected earning is $E(h) = hp^h$, which gives
   $$E(h) = \frac{hN^h}{(N+1)^h}.$$

   Notice that
   $$\frac{E(h+1)}{E(h)} = \frac{(h+1)N^{h+1}/(N+1)^{h+1}}{hN^h/(N+1)^h} = \frac{(h+1)N}{h(N+1)}.$$

   We have
   $$\frac{E(h+1)}{E(h)} - 1 = \frac{(hN+N)-(hN+h)}{hN+h} = \frac{N-h}{hN+h},$$

   which shows that $E(h+1) > E(h)$ when $h < N$, and $E(h+1) < E(h)$ when $h > N$, and $E(h+1) = E(h)$ when $h = N$.

   This means that $E(h)$ will increase until $h = N$, where $E(N) = E(N+1)$, and decrease after $h = N+1$.

   This means the expected earnings can be maximised when $h = N$ or $h = N+1$, which shows when $h = N$, the earnings is maximised.

2. There are two cases: either the person earns £h (when there are h heads thrown before the game ends) with some probability (that we would like to find), or the game ends before there are h heads thrown.

   To find the probability in the first case, let there be t cases where a tail appears, and there must be h cases where a head appears. The final throw must be a head, and the tail must appear singularly (which means any two consecutive tails must have a head in between), which shows that $0 \leq t \leq h$.

   There are $h-1$ heads that are free to 'move', and t tails have $t-1$ gaps in between, which takes away at least $t-1$ heads to separate them. The rest of the $h-t$ heads are free to be within any of the $t+1$ spaces that are separated by the t tails, which is equivalent of choosing t to be heads from a total $(h-t)+t = h$ remaining throws.

   Therefore, for each t, the number of arrangements there are is
   $$\binom{h}{t},$$

   and the probability of this happening is
   $$p^h \cdot (1-p)^t.$$

   Therefore, the probability desired is
   $$\sum_{t=0}^{h} \binom{h}{t} p^h (1-p)^t = p^h \sum_{t=0}^{h} \binom{h}{t} 1^{h-t}(1-p)^t = p^h(1+1-p)^h = p^h(2-p)^h,$$

   and the expected earnings in terms of h is
   $$E(h) = hp^h(2-p)^h = h\left(\frac{N}{N+1}\right)^h \left(\frac{N+2}{N+1}\right)^h = \frac{hN^h(N+2)^h}{(N+1)^{2h}}$$

   as desired.

   When $N = 2$,
   $$E(h) = \frac{h2^h4^h}{3^{2h}} = \frac{h2^{3h}}{3^{2h}}.$$

Notice that
$$\frac{E(h+1)}{E(h)} = \frac{(h+1)2^{3h+3}/3^{2h+2}}{h2^{3h}/3^{2h}} = \frac{8(h+1)}{9h},$$
and hence
$$\frac{E(h+1)}{E(h)} - 1 = \frac{8-h}{9h},$$
which shows that $E(h+1) > E(h)$ when $h < 8$, and $E(h+1) < E(h)$ when $h > 8$, and $E(h+1) = E(h)$ when $h = 8$.

This shows that $E(8) = E(9)$ gives the maximum expected winnings, which is given by
$$\frac{8 \cdot 2^{24}}{3^{16}} = \frac{2^{27}}{3^{16}}.$$

Since $\log_3 2 \approx 0.63$, we have $2 \approx 3^{0.63}$, and hence
$$\frac{2^{27}}{3^{16}} \approx \frac{3^{27 \cdot 0.63}}{3^{16}} = 3^{27 \cdot 0.63 - 16} = 3^{1.01} \approx 3,$$
and this shows that the maximum value of expected winnings is approximately £3.

2018.2 Question 13

This setup gives a Markov Chain. Let the column vector $\mathbf{x}_n$ represent a state

$$\mathbf{x}_n = \begin{pmatrix} A_n \\ B_n \\ C_n \\ D_n \end{pmatrix},$$

and hence we have the components of the column vector must sum to 1. The initial state is defined by

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and the state transition matrix is

$$\mathbf{M} = \begin{pmatrix} 1/2 & 1/4 & 0 & 1/4 \\ 1/4 & 1/2 & 1/4 & 0 \\ 0 & 1/4 & 1/2 & 1/4 \\ 1/4 & 0 & 1/4 & 1/2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix},$$

which gives

$$\mathbf{x}_{n+1} = \mathbf{M}\mathbf{x}_n.$$

1. Notice that

$$\mathbf{x}_1 = \mathbf{M}\mathbf{x}_0 = \frac{1}{4}\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix},$$

   and hence $A_1 = \frac{1}{2}, B_1 = \frac{1}{4}, C_1 = 0, D_1 = \frac{1}{4}$.
   Also,

$$\mathbf{x}_2 = \mathbf{M}\mathbf{x}_1 = \frac{1}{4}\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix} \cdot \frac{1}{4}\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{16}\begin{pmatrix} 6 \\ 4 \\ 2 \\ 4 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \end{pmatrix},$$

   and hence $A_2 = \frac{3}{8}, B_2 = \frac{1}{4}, C_2 = \frac{1}{8}, D_2 = \frac{1}{4}$.

2. We claim that $B_n = D_n$ for all n by symmetry, and notice that

$$B_{n+1} = \frac{1}{4} \cdot (A_n + 2B_n + C_n) = \frac{1}{4} \cdot (A_n + B_n + C_n + D_n) = \frac{1}{4},$$

   and

$$D_{n+1} = \frac{1}{4} \cdot (A_n + C_n + 2D_n) = \frac{1}{4} \cdot (A_n + B_n + C_n + D_n) = \frac{1}{4},$$

   so that $B_n = D_n = \frac{1}{4}$ for all $n \geq 1$. (For $n = 0$, $B_n = D_n = 0$).
   Hence, for $n \geq 1$, we have

$$A_{n+1} = \frac{1}{4}(2A_n + B_n + D_n) = \frac{1}{4}\left(2A_n + \frac{1}{2}\right) = \frac{1}{2}A_n + \frac{1}{8},$$

   which means

$$A_{n+1} - \frac{1}{4} = \frac{1}{2}\left(A_n - \frac{1}{4}\right),$$

   which shows that $A_n - \frac{1}{4}$ is a geometric sequence with common ratio $\frac{1}{2}$. The initial term of the geometric sequence is $A_1 - \frac{1}{4} = \frac{1}{4}$, and hence

$$A_n - \frac{1}{4} = \frac{1}{2^{n+1}},$$

which shows $A_n = \frac{1}{4} + \frac{1}{2^{n+1}}$ for $n \geq 1$.

Also, C_n has the same inductive relationship as A_n, the only difference being that the initial term is $C_1 - \frac{1}{4} = -\frac{1}{4}$, and hence

$$C_n - \frac{1}{4} = -\frac{1}{2^{n+1}},$$

which shows $C_n = \frac{1}{4} - \frac{1}{2^{n+1}}$ for $n \geq 1$.

Hence, we have

$$\mathbf{x}_n = \begin{pmatrix} A_n \\ B_n \\ C_n \\ D_n \end{pmatrix} = \begin{cases} (1, 0, 0, 0)^\mathsf{T}, & n = 0, \\ \left(\frac{1}{4} + \frac{1}{2^{n+1}}, \frac{1}{4}, \frac{1}{4} - \frac{1}{2^{n+1}}, \frac{1}{4}\right)^\mathsf{T}, & \text{otherwise.} \end{cases}$$

2018 Paper 3

2018.3 Question 1

1. By differentiation with respect to β, we have

$$f'(\beta) = 1 + \frac{1}{\beta^2} + \frac{2}{\beta^3}.$$

If $f'(t) = 0$, we must have

$$t^3 + t + 2 = 0.$$

Therefore,

$$(t+1)(t^2 - t + 2) = 0,$$

and hence the only real root to this is $t = -1$, since $(-1)^2 - 2 \cdot 4 < 0$.

This means the only stationary point of $y = f(\beta)$ is $(-1, f(-1) = -1)$.

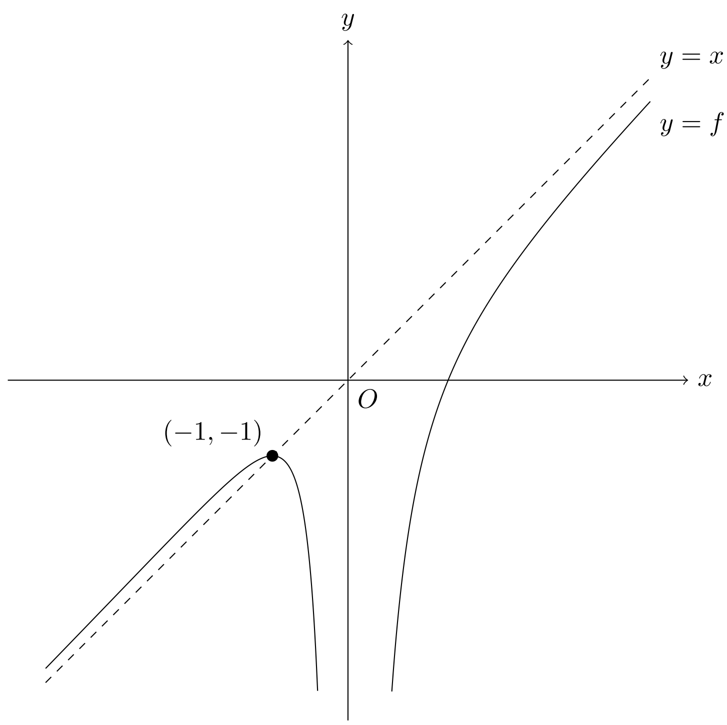
For the limiting behaviour of the function, we first look at the case where $\beta > 0$. As $\beta \to \infty$, we have $f(\beta) \to \beta$ from below. As $\beta \to 0^+$, we have $f(\beta) \to -\frac{1}{\beta} - \frac{1}{\beta^2} \to -\infty$.

When $\beta < 0$, we use the substitution $t = -\frac{1}{\beta}$ to make the behaviours more convincing, and hence

$$f(\beta) = \beta + t - t^2.$$

As $\beta \to 0^-$, we have $t \to \infty$, and $f(\beta) \to t - t^2 \to -\infty$. As $\beta \to -\infty$, we have $t \to 0^+$, and $f(\beta) \to \beta$ from above, since $t - t^2 = t(1-t) > 0$ when $0 < t < 1$.

This means the curve $y = f(\beta)$ is as below.



Similarly, by differentiation with respect to β, we have

$$g'(\beta) = 1 - \frac{3}{\beta^2} + \frac{2}{\beta^3}.$$

If $g'(t) = 0$, we must have

$$t^3 - 3t + 2 = 0.$$

Therefore,

$$(t-1)^2(t+2) = 0,$$

and hence the real roots to this is $t = 1$ and $t = -2$.

This means the stationary points of $y = g(\beta)$ is $(1, g(1) = 3)$ and $(-2, g(-2) = -\frac{15}{4})$.

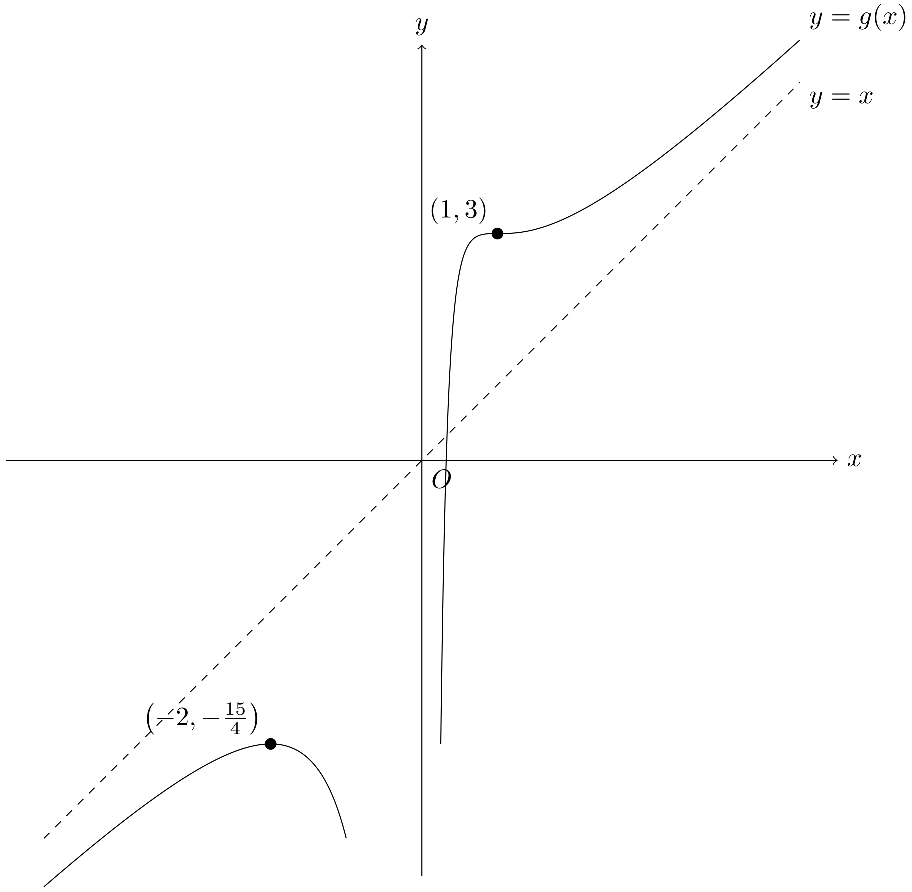
For the limiting behaviour of the function, we first look at the case where $\beta > 0$. We consider the substitution $t = -\frac{1}{\beta}$ to make the behaviours more convincing, and hence

$$g(\beta) = \beta - 3t - t^2.$$

As $\beta \to \infty$, $t \to 0^-$, and hence $f(\beta) \to \beta$ from below, since $-3t - t^2 = -t(t+3) > 0$ for $-3 < t < 0$. As $\beta \to 0^+$, $t \to -\infty$, and hence $f(\beta) \to -3t - t^2 \to -\infty$.

When $\beta < 0$, we have as $\beta \to 0^-$, $f(\beta) \to -\infty$. As $\beta \to -\infty$, $f(\beta) \to \beta$ from below.

This means the curve $y = g(\beta)$ is as below.



2. By Vieta's Theorem, we have $u + v = -\alpha$, and $uv = \beta$. Hence,

$$u + v + \frac{1}{uv} = -\alpha + \frac{1}{\beta},$$

and

$$\frac{1}{u} + \frac{1}{v} + uv = \frac{u+v}{uv} + uv = -\frac{\alpha}{\beta} + \beta.$$

3. By the given condition, we have

$$-\alpha + \frac{1}{\beta} = -1 \iff \alpha = 1 + \frac{1}{\beta}.$$

Hence,

$$\begin{aligned}\frac{1}{u}+\frac{1}{v}+uv &= -\frac{\alpha}{\beta}+\beta \\ &= -\frac{1+\frac{1}{\beta}}{\beta}+\beta \\ &= \frac{\beta^2-1-\frac{1}{\beta}}{\beta} \\ &= \beta-\frac{1}{\beta}-\frac{1}{\beta^2} \\ &= f(\beta).\end{aligned}$$

Also, since u, v are both real, we have

$$\begin{aligned}\alpha^2-4\beta &= \left(1+\frac{1}{\beta}\right)^2-4\beta \\ &= 1+\frac{2}{\beta}+\frac{1}{\beta^2}-4\beta \\ &= \frac{-4\beta^3+\beta^2+2\beta+1}{\beta^2} \\ &\geq 0.\end{aligned}$$

Multiplying both sides by $-\beta^2$ (which flips the sign) gives

$$\begin{aligned}4\beta^3-\beta^2-2\beta-1\leq 0 \\ (\beta-1)(4\beta^2+3\beta+1)\leq 0.\end{aligned}$$

This cubic has exactly one real root $\beta = 1$, so the solution to this inequality is $\beta \leq 1$ and $\beta \neq 0$. Notice that f is increasing on $(0, 1] \subset (0, \infty)$. Therefore, for $\beta > 0$,

$$f(\beta) \leq f(1) = 1-1-1 = -1.$$

When $\beta < 0$, we have

$$f(\beta) \leq f(-1) = -1.$$

So for the range of β in this question, we always have $f(\beta) \leq -1$. But we also have $\frac{1}{u}+\frac{1}{v}+uv \leq -1$ as shown before. These gives us exactly our desired statement.

4. By the given condition, we have

$$-\alpha+\frac{1}{\beta} = 3 \iff \alpha = -3+\frac{1}{\beta}.$$

Hence,

$$\begin{aligned}\frac{1}{u}+\frac{1}{v}+uv &= -\frac{\alpha}{\beta}+\beta \\ &= -\frac{-3+\frac{1}{\beta}}{\beta}+\beta \\ &= \beta+\frac{3}{\beta}-\frac{1}{\beta^2} \\ &= g(\beta).\end{aligned}$$

Also, since u, v are both real, we have $\beta \leq 1$ and $\beta \neq 0$ as well.

g must be increasing on $(0, 1]$. Hence, for $\beta > 0$, we have

$$g(\beta) \leq g(1) = 3.$$

When $\beta < 0$, we have

$$g(\beta) \leq g(-2) = -\frac{15}{4}.$$

Since $3 > -\frac{15}{4}$, we can conclude that the maximum value of $\frac{1}{u} + \frac{1}{v} + uv$ is 3, and it is taken when $\beta = 1$, which corresponds to $\alpha = -2$.

2018.3 Question 2

1. Notice that

$$\begin{aligned}
\frac{\mathrm{d}y_n}{\mathrm{d}x} &= \frac{\mathrm{d}(-1)^n \frac{1}{z} \frac{\mathrm{d}^n z}{\mathrm{d}x^n}}{\mathrm{d}x} \\
&= (-1)^n \left[\frac{\mathrm{d}\frac{1}{z}}{\mathrm{d}x} \cdot \frac{\mathrm{d}^n z}{\mathrm{d}x^n} + \frac{1}{z} \cdot \frac{\mathrm{d}\frac{\mathrm{d}^n z}{\mathrm{d}x^n}}{\mathrm{d}x} \right] \\
&= (-1)^n \left[\frac{2x}{z} \cdot \frac{\mathrm{d}^n z}{\mathrm{d}x^n} + \frac{1}{z} \cdot \frac{\mathrm{d}^{n+1} z}{\mathrm{d}x^{n+1}} \right] \\
&= 2x \cdot (-1)^n \frac{1}{z} \frac{\mathrm{d}^n z}{\mathrm{d}x^n} - (-1)^{n+1} \frac{1}{z} \frac{\mathrm{d}^{n+1} z}{\mathrm{d}x^{n+1}} \\
&= 2xy_n - y_{n+1},
\end{aligned}$$

as desired.

2. We first look at the base case where $n = 1$. What is desired is

$$y_2 = 2xy_1 - 2y_0.$$

We have $y_0 = 1$,

$$y_1 = (-1)^1 \frac{1}{e^{-x^2}} \frac{\mathrm{d}e^{-x^2}}{\mathrm{d}x} = -e^{x^2}(-2x)e^{-x^2} = 2x,$$

and

$$y_2 = 2xy_1 - \frac{\mathrm{d}y_1}{\mathrm{d}x} = 2x \cdot 2x - 2 = 4x^2 - 2.$$

Hence,

$$2xy_1 - 2y_0 = 2x \cdot 2x - 2 \cdot 1 - 4x^2 - 2 = y_2,$$

so the base case is satisfied.

Now assume this is true for some $n = k \geq 1$, i.e.

$$y_{k+1} = 2xy_k - 2ky_{k-1}.$$

We have

$$\begin{aligned}
y_{k+2} &= 2xy_{k+1} - \frac{\mathrm{d}y_{k+1}}{\mathrm{d}x} \\
&= 2xy_{k+1} - \frac{\mathrm{d}\left(2xy_k - 2ky_{k-1}\right)}{\mathrm{d}x} \\
&= 2xy_{k+1} - 2y_k - 2x\frac{\mathrm{d}y_k}{\mathrm{d}x} + 2k\frac{\mathrm{d}y_{k-1}}{\mathrm{d}x} \\
&= 2xy_{k+1} - 2y_k - 2x(2xy_k - y_{k+1}) + 2k(2xy_{k-1} - y_k) \\
&= 2xy_{k+1} - 2y_k - 4x^2y_k + 2xy_{k+1} + 4kxy_{k-1} - 2ky_k \\
&= 4xy_{k+1} - 2(2x^2 + k + 1)y_k + 4kxy_{k-1} \\
&= 4xy_{k+1} - 2(2x^2 + k + 1)y_k + 4kx \cdot \frac{2xy_k - y_{k+1}}{2k} \\
&= 4xy_{k+1} - 2(2x^2 + k + 1)y_k + 2x(2xy_k - y_{k+1}) \\
&= 2xy_{k+1} - 2(k+1)y_k,
\end{aligned}$$

which is exactly the statement for $n = k + 1$.

Hence, by the principle of mathematical induction, we have $y_{n+1} = 2xy_n - 2ny_{n-1}$ for all $n \geq 1$.

We have

$$\begin{aligned}
\text{LHS} &= y_{n+1}^2 - y_n y_{n+2} \\
&= y_{n+1}^2 - y_n(2xy_{n+1} - 2(n+1)y_n) \\
&= y_{n+1}^2 - 2xy_n y_{n+1} + 2(n+1)y_n^2
\end{aligned}$$

and

$$\begin{aligned}
\text{RHS} &= 2n(y_n^2 - y_{n-1}y_{n+1}) + 2y_n^2 \\
&= 2n\left(y_n^2 - \frac{2xy_n - y_{n+1}}{2n}y_{n+1}\right) + 2y_n^2 \\
&= 2ny_n^2 - (2xy_n - y_{n+1})\,y_{n+1} + 2y_n^2 \\
&= 2ny_n^2 - 2xy_ny_{n+1} + y_{n+1}^2 + 2y_n^2 \\
&= y_{n+1}^2 - 2xy_ny_{n+1} + 2(n+1)y_n^2.
\end{aligned}$$

3. This can be shown by induction on n. The base case for $n = 1$ is

$$y_1^2 - y_0y_2 = (2x)^2 - 1 \cdot (4x^2 - 2) = 2 > 0$$

is true.

Now assume the statement is true for $n = k \geq 1$, i.e.

$$y_k^2 - y_{k-1}y_{k+1} > 0.$$

We have

$$\begin{aligned}
y_{k+1}^2 - y_ky_{k+2} &= 2n(y_k^2 - y_{k-1}y_k + 1) + 2y_n^2 \\
&> 2n \cdot 0 + y_n^2 \\
&= 0 + y_n^2 \\
&\geq 0,
\end{aligned}$$

which is the statement for $n = k + 1$.

Hence, by the principle of mathematical induction, we have $y_n^2 - y_{n-1}y_{n+1} > 0$ for all $n \geq 1$.

2018.3 Question 3

Notice that

$$\begin{aligned}
x^a(x^b(x^cy)')' &= x^a(x^b(cx^{c-1}y+x^cy'))' \\
&= x^a\left[x^{b+c-1}\left(cy+xy'\right)\right]' \\
&= x^a\left[(b+c-1)x^{b+c-2}\left(cy+xy'\right)+x^{b+c-1}\left(cy'+y'+xy''\right)\right] \\
&= x^{a+b+c-2}\left[(b+c-1)\left(cy+xy'\right)+x\left(cy'+y'+xy''\right)\right] \\
&= x^{a+b+c-2}\left[x^2y''+(b+2c)xy'+(b+c-1)cy\right].
\end{aligned}$$

Comparing this with the left-hand side of the original equation, we must have

$$\begin{cases} a+b+c-2=0, \\ b+2c=1-2p, \\ (b+c-1)c=p^2-q^2. \end{cases}$$

The second equation gives

$$b=1-2p-2c,$$

and putting this into the third equation gives

$$(1-2p-2c+c-1)c=p^2-q^2,$$

and hence

$$c^2+2pc+p^2-q^2=0.$$

This gives

$$(c+(p-q))(c+(p+q))=0,$$

and hence

$$c_1=-p+q, c_2=-p-q.$$

Putting this back, we get

$$b_1=1-2p-2(-p+q)=1-2q, b_2=1-2p-2(-p-q)=1+2q,$$

and since $a=2-b-c$ from the first equation, we have

$$a_1=2-(1-2q)-(-p+q)=1+p+q$$

and

$$a_2=2-(1+2q)-(-p-q)=1+p-q$$

Hence, the solutions are

$$\begin{cases} a=p\pm q+1, \\ b=\mp 2q+1, \\ c=-p\pm q. \end{cases}$$

1. In the case where $f(x)=0$. We must have

   $$x^a\left(x^b(x^cy)'\right)'=0,$$

   and hence

   $$\left(x^b(x^cy)'\right)'=0.$$

   Therefore, we must have by integration

   $$x^b(x^cy)'=C_1$$

   for some (real) constant C_1.
   Hence,

   $$(x^cy)'=C_1x^{-b}.$$

   There are two cases here:

(a) When $b = 1$ i.e. $q = 0$, the right-hand side is $C_1 x^{-1}$, and the left-hand side is $(x^c y)'$. Integrating both sides give

$$x^c y = C_1 \ln x + C_2$$

for some (real) constant C_2.
Hence,

$$y = x^{-c}(C_1 \ln x + C_2)$$

for some (real) constants C_1, C_2.
When $q = 0$, $c = -p$, and hence

$$y = x^p(C_1 \ln x + C_2).$$

(b) When $b \neq 1$ i.e. $q \neq 0$, integrating both sides give

$$x^c y = \frac{C_1 x^{-b+1}}{-b+1} + C_2$$

for some (real) constant C_2.
Hence,

$$y = x^{-c}\left(\frac{C_1 x^{-b+1}}{-b+1} + C_2\right)$$

for some (real) constant C_1, C_2.
Hence,

$$\begin{aligned} y &= x^{-(-p \pm q)}\left(\frac{C_1 x^{-(\mp 2q+1)+1}}{-(\mp 2q+1)+1} + C_2\right) \\ &= x^{p \mp q}\left(\frac{C_1 x^{\pm 2q}}{\pm 2q} + C_2\right). \\ &= \frac{C_1}{\pm 2q} x^{p \pm q} + C_2 x^{p \mp q} \\ &= C_3 x^{p \pm q} + C_2 x^{p \mp q}, \end{aligned}$$

for some (real) constant C_2, C_3.

2. This is when $q = 0$ and $f(x) = x^n$. We have $a = p+1, b = 1$ and $c = -p$, and the original differential equation reduces to

$$x^{p+1}\left(x\left(x^{-p}y\right)'\right)' = x^n,$$

and hence

$$\left(x\left(x^{-p}y\right)'\right)' = x^{n-p-1}.$$

There are two cases here:

(a) If $n - p - 1 = -1$, i.e. $n = p$, we have, by integration,

$$x\left(x^{-p}y\right)' = \ln x + C_1.$$

This gives

$$\left(x^{-p}y\right)' = \frac{\ln x}{x} + \frac{C_1}{x},$$

and hence by integration

$$x^{-p}y = \frac{(\ln x)^2}{2} + C_1 \ln x + C_2.$$

This solves to

$$y = \frac{x^p (\ln x)^2}{2} + C_1 x^p \ln x + C_2 x^p.$$

(b) If $n - p - 1 \neq -1$, i.e. $n \neq p$, we have

$$x\left(x^{-p}y\right)' = \frac{x^{n-p}}{n-p} + C_1.$$

This gives

$$\left(x^{-p}y\right)' = \frac{x^{n-p-1}}{n-p} + \frac{C_1}{x}.$$

Since $n - p - 1 \neq -1$, by integration we have

$$x^{-p}y = \frac{x^{n-p}}{(n-p)^2} + C_1 \ln x + C_2,$$

and hence

$$y = \frac{x^n}{(n-p)^2} + C_1 x^p \ln x + C_2 x^p.$$

2018.3 Question 4

The hyperbola has parametric equation

$$\begin{cases} x = a\sec\theta, \\ y = b\tan\theta. \end{cases}$$

Hence, by differentiation, we have

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} \\ &= \frac{b\sec^2\theta}{a\sec\theta\tan\theta} \\ &= \frac{b\cos\theta}{a\sin\theta\cos\theta} \\ &= \frac{b}{a\sin\theta}. \end{aligned}$$

The tangent to the hyperbola at P will be

$$y - b\tan\theta = \frac{b}{a\sin\theta}(x - a\sec\theta),$$

which simplifies to

$$ay\sin\theta - ab\tan\theta\sin\theta = bx - ab\sec\theta,$$

and hence

$$bx - ay\sin\theta = ab(\sec\theta - \tan\theta\sin\theta).$$

Notice that

$$\sec\theta - \tan\theta\sin\theta = \frac{1-\sin^2\theta}{\cos\theta} = \frac{\cos^2\theta}{\cos\theta} = \cos\theta,$$

and so the equation of the tangent is

$$bx - ay\sin\theta = ab\cos\theta,$$

exactly as desired.

1. Let $\frac{x}{a} = \frac{y}{b} = s$ for S, we have $x = as$ and $y = bs$, and hence

   $$abs - abs\sin\theta = ab\cos\theta,$$

   which gives

   $$s = \frac{\cos\theta}{1-\sin\theta},$$

   and hence

   $$S\left(a\frac{\cos\theta}{1-\sin\theta}, b\frac{\cos\theta}{1-\sin\theta}\right).$$

   Let $\frac{x}{a} = -\frac{y}{b} = t$ for T, we have $x = at$ and $y = -bt$, and hence

   $$abt + abt\sin\theta = ab\cos\theta,$$

   which gives

   $$t = \frac{\cos\theta}{1+\sin\theta},$$

   and hence

   $$T\left(a\frac{\cos\theta}{1+\sin\theta}, -b\frac{\cos\theta}{1+\sin\theta}\right).$$

We have

$$\begin{aligned}\frac{a\frac{\cos\theta}{1-\sin\theta}+a\frac{\cos\theta}{1+\sin\theta}}{2}&=\frac{a\cos\theta}{2}\left(\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}\right)\\&=\frac{a\cos\theta}{2}\left(\frac{2}{\cos^2\theta}\right)\\&=\frac{a}{\cos\theta}\\&=a\sec\theta,\end{aligned}$$

and

$$\begin{aligned}\frac{a\frac{\cos\theta}{1-\sin\theta}-b\frac{\cos\theta}{1+\sin\theta}}{2}&=\frac{b\cos\theta}{2}\left(\frac{1}{1-\sin\theta}-\frac{1}{1+\sin\theta}\right)\\&=\frac{b\cos\theta}{2}\left(\frac{2\sin\theta}{\cos^2\theta}\right)\\&=\frac{b\sin\theta}{\cos\theta}\\&=b\tan\theta.\end{aligned}$$

This means the midpoint of ST is $(a\sec\theta, b\tan\theta)$, which is exactly P.

2. Since the tangents are perpendicular, that means

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_\theta\cdot\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_\varphi=-1,$$

and hence

$$\frac{b}{a\sin\theta}\cdot\frac{b}{a\sin\varphi}=-1,$$

which means

$$b^2=-a^2\sin\theta\sin\varphi.$$

The two tangents are

$$bx-ay\sin\theta=ab\cos\theta$$

and

$$bx-ay\sin\varphi=ab\cos\varphi.$$

Since $bx=bx$, we have

$$ay\sin\theta+ab\cos\theta=ay\sin\varphi+ab\cos\varphi,$$

and hence

$$y(\sin\theta-\sin\varphi)=b(\cos\varphi-\cos\theta),$$

which gives

$$y=b\cdot\frac{\cos\varphi-\cos\theta}{\sin\theta-\sin\varphi}.$$

Hence,

$$\begin{aligned}x&=\frac{ab\cos\theta+ay\sin\theta}{b}\\&=\frac{a}{b}\left(b\cos\theta+b\sin\theta\frac{\cos\varphi-\cos\theta}{\sin\theta-\sin\varphi}\right)\\&=a\left(\cos\theta+\sin\theta\frac{\cos\varphi-\cos\theta}{\sin\theta-\sin\varphi}\right)\\&=a\frac{\cos\theta(\sin\theta-\sin\varphi)+\sin\theta(\cos\varphi-\cos\theta)}{\sin\theta-\sin\varphi}\\&=a\cdot\frac{\sin\theta\cos\varphi-\cos\theta\sin\varphi}{\sin\theta-\sin\varphi}\\&=a\cdot\frac{\sin(\theta-\varphi)}{\sin\theta-\sin\varphi}.\end{aligned}$$

This means

$$\begin{cases} x^2 = a^2 \cdot \dfrac{\sin^2(\theta - \varphi)}{(\sin\theta - \sin\varphi)^2}, \\ y^2 = b^2 \cdot \dfrac{(\cos\varphi - \cos\theta)^2}{(\sin\theta - \sin\varphi)^2} = -a^2 \sin\theta \sin\varphi \cdot \dfrac{(\cos\varphi - \cos\theta)^2}{(\sin\theta - \sin\varphi)^2}. \end{cases}$$

Notice that

$$a^2 - b^2 = a^2 + a^2 \sin\theta \sin\varphi = a^2(1 + \sin\theta \sin\varphi).$$

Hence,

$$\begin{aligned} x^2 + y^2 &= a^2 \left[\frac{\sin^2(\theta - \varphi)}{(\sin\theta - \sin\varphi)^2} - \sin\theta \sin\varphi \cdot \frac{(\cos\varphi - \cos\theta)^2}{(\sin\theta - \sin\varphi)^2} \right] \\ &= \frac{a^2}{(\sin\theta - \sin\varphi)^2} \left[\sin^2(\theta - \varphi) - \sin\theta \sin\varphi \left(\cos\varphi - \cos\theta\right)^2 \right]. \end{aligned}$$

What is desired is to show

$$(1 + \sin\theta \sin\varphi)(\sin\theta - \sin\varphi)^2 = \sin^2(\theta - \varphi) - \sin\theta \sin\varphi \left(\cos\varphi - \cos\theta\right)^2.$$

We have

$$\begin{aligned} \text{RHS} &= (\sin\theta \cos\varphi - \cos\theta \sin\varphi)^2 - \sin\theta \sin\varphi (\cos^2\varphi + \cos^2\theta - 2\cos\varphi \cos\theta) \\ &= \sin^2\theta \cos^2\varphi + \cos^2\theta \sin^2\varphi - 2\sin\theta \cos\theta \sin\varphi \cos\varphi \\ &\quad - \sin\theta \sin\varphi \cos^2\varphi - \sin\theta \sin\varphi \cos^2\theta + 2\sin\theta \cos\theta \sin\varphi \cos\varphi \\ &= \sin\theta \cos^2\varphi (\sin\theta - \sin\varphi) + \cos^2\theta \sin\varphi (\sin\varphi - \sin\theta) \\ &= (\sin\theta \cos^2\varphi - \cos^2\theta \sin\varphi)(\sin\theta - \sin\varphi). \end{aligned}$$

Therefore, what is left to prove is that

$$(1 + \sin\theta \sin\varphi)(\sin\theta - \sin\varphi) = \sin\theta \cos^2\varphi - \cos^2\theta \sin\varphi$$

Notice that

$$\begin{aligned} \text{LHS} &= \sin\theta - \sin\varphi + \sin^2\theta \sin\varphi - \sin\theta \sin^2\varphi \\ &= \sin\theta(1 - \sin^2\varphi) - \sin\varphi(1 - \sin^2\theta) \\ &= \sin\theta \cos^2\varphi - \sin\varphi \cos^2\theta \\ &= \text{RHS}. \end{aligned}$$

This shows that

$$\frac{1}{(\sin\theta - \sin\varphi)^2} \left[\sin^2(\theta - \varphi) - \sin\theta \sin\varphi \left(\cos\varphi - \cos\theta\right)^2 \right] = 1 + \sin\theta \sin\varphi,$$

and hence

$$x^2 + y^2 = a^2 - b^2,$$

as desired.

2018.3 Question 5

1. First, we notice that
$$G_{k+1}^{k+1} = \prod_{t=1}^{k+1} a_t = a_{k+1} G_k^k,$$
and hence
$$G_{k+1} = \left(a_{k+1} G_k^k\right)^{\frac{1}{k+1}}.$$
Similarly, notice that
$$(k+1)A_{k+1} = \sum_{t=1}^{k+1} a_t = a_{k+1} + kA_k.$$
Hence,
$$\begin{aligned} (k+1)\left(A_{k+1} - G_{k+1}\right) &\geq k\left(A_k - G_k\right), \\ a_{k+1} + kA_k - (k+1)\left(a_{k+1} G_k^k\right)^{\frac{1}{k+1}} &\geq ka_k - kG_k, \\ a_{k+1} + kG_k &\geq (k+1) a_{k+1}^{\frac{1}{k+1}} G_k^{\frac{k}{k+1}}. \end{aligned}$$
Dividing both sides by G_k, we have
$$\begin{aligned} \frac{a_{k+1}}{G_k} + k &\geq (k+1) a_{k+1}^{\frac{1}{k+1}} G_k^{-\frac{1}{k+1}}, \\ \lambda_k^{k+1} + k &\geq (k+1)\left(\frac{a_{k+1}}{G_k}\right)^{\frac{1}{k+1}}, \\ \lambda_k^{k+1} + k &\geq (k+1)\lambda_k, \\ \lambda_k^{k+1} - (k+1)\lambda_k + k &\geq 0, \end{aligned}$$
as desired. (Notice that the condition for the equal sign is equivalent as well.)

2. By differentiation, we have
$$f'(x) = (k+1)x^k - (k+1) = (k+1)(x^k - 1).$$
When $x \in (0,1), x^k \in (0,1), f'(x) < 0$, and hence f is strictly decreasing.
When $x \in (1,\infty), x^k \in (1,\infty), f'(x) > 0$, and hence f is strictly increasing.
Hence, $f(1)$ is the minimum for f on $(0,\infty)$. This means for all $x \in (0,\infty)$, we have
$$f(x) \geq f(1) = 1^{k+1} - (k+1) + k = 0,$$
taking the equal sign if and only if $x = 1$.

3. (a) We show this by induction. For the base case $n = 1$, $A_1 = G_1 = a_1$, so naturally $A_n \geq G_n$ is satisfied.
Assume that the statement holds for some $n = k$, i.e. $A_k \geq G_k$, $A_k - G_k \geq 0$. Since $k > 0$ as well, we must have
$$(k+1)(A_{k+1} - G_{k+1}) \geq k(A_k - G_k) \geq 0.$$
We also have $k + 1 > 0$, and hence
$$A_{k+1} - G_{k+1} \geq 0 \iff A_{k+1} \geq G_{k+1},$$
meaning the statement holds for $n = k + 1$ as well.
Hence, by the principle of mathematical induction, we must have $A_n \geq G_n$ for all $n \in \mathbb{N}$, which finishes our proof.

(b) We show this by induction. For the base case $n = 1$, this condition is naturally satisfied.

Assume that the statement holds for some $n = k$, i.e. $A_k = G_k \implies a_1 = a_2 = \cdots = a_k$. We show this for $n = k + 1$. If $A_{k+1} = G_{k+1}$, then we must have

$$k(A_k - G_k) \leq (k+1)(A_{k+1} - G_{k+1}) = 0,$$

but since $A_k \geq G_k$, we must have then $A_k = G_k$, and hence the equal sign in the inequality being taken.

This must mean that

$$\lambda_k = \left(\frac{a_{k+1}}{G_k}\right)^{\frac{1}{k+1}} = 1,$$

and hence

$$a_{k+1} = G_k.$$

At the same time, since $A_k = G_k$, we must have $a_1 = a_2 = \cdots = a_k$, and hence $G_k = a_1 = a_2 = \cdots = a_k$. Therefore, we must also have

$$a_1 = a_2 = \cdots = a_k = a_{k+1},$$

which proves the statement that $A_{k+1} = G_{k+1}$ implies $a_1 = a_2 = \cdots = a_k = a_{k+1}$, which is the original statement for $n = k + 1$.

Hence, by the principle of mathematical induction, we must have $A_n = G_n$ implies $a_1 = a_2 = \cdots = a_n$ for all $n \in \mathbb{N}$, which finishes our proof.

2018.3 Question 6

1. Since A, Q, C lie on a straight line, $\mathbf{AQ} = \lambda \mathbf{AC}$ for some $\lambda \in \mathbb{R}$. This means

$$q - a = \lambda(c - a),$$

and hence

$$\frac{q-a}{c-a} = \lambda \in \mathbb{R},$$

as required.

Hence, we must have

$$\frac{q-a}{c-a} = \left(\frac{q-a}{c-a}\right)^* = \frac{q^*-a^*}{c^*-a^*}.$$

Cross-multiplying the terms out give

$$(c-a)(q^*-a^*) = (c^*-a^*)(q-a)$$

exactly as desired.

Substituting in $a^* = 1/a$ and $c^* = 1/c$, we have

$$(c-a)\left(q^* - \frac{1}{a}\right) = \left(\frac{1}{c} - \frac{1}{a}\right)(q-a),$$

and expanding the brackets gives

$$cq^* - aq^* - \frac{c}{a} + 1 = \frac{q}{c} - \frac{a}{c} - \frac{q}{a} + 1,$$

and hence

$$cq^* - aq^* - \frac{c}{a} = \frac{q}{c} - \frac{a}{c} - \frac{q}{a}.$$

Multiplying by ac on both sides gives us

$$ac^2q^* - a^2cq^* - c^2 = aq - a^2 - cq,$$

and hence

$$ac(c-a)q^* = (a-c)q - (a^2-c^2) = (a-c)q - (a-c)(a+c).$$

We can divide through $(a-c)$ on both sides since $a \neq c$. Hence,

$$0 = q - (a+c) + acq^*,$$

and hence

$$acq^* + q = a + c,$$

as desired.

2. By part 1, we must have

$$acq^* + q = a + c, bdq^* + q = b + d.$$

Since $q = q$, we have

$$acq^* - (a+c) = bdq^* - (b+d),$$

and rearranging gives

$$(ac - bd)q^* = (a+c) - (b+d),$$

exactly as desired.

We also have $q^* = q^*$, and hence

$$\frac{a+c-q}{ac} = \frac{b+d-q}{bd},$$

which gives

$$(bd)(a+c-q) = (ac)(b+d-q),$$

and rearranging gives

$$(ac - bd)q = ac(b + d) - bd(a + c).$$

Summing this with previously, we have

$$(ac - bd)(q + q^*) = (a + c) - (b + d) + ac(b + d) - bd(a + c).$$

We notice that

$$\begin{aligned}(a + c) - (b + d) + ac(b + d) - bd(a + c) &= a + c - b - d + abc + acd - abd - bcd \\ &= a - b + acd - bcd + c - d + abc - abd \\ &= (a - b)(1 + cd) + (c - d)(1 + ab),\end{aligned}$$

and hence

$$(ac - bd)(q + q^*) = (a - b)(1 + cd) + (c - d)(1 + ab),$$

exactly as desired.

3. By part 1, we must have

$$p + abp^* = a + b.$$

Since p is real, $p = p^*$, and hence

$$(1 + ab)p = a + b,$$

as desired.

Similarly, we must have

$$(1 + cd)q = c + d,$$

and putting this back into the result from part 2, we have

$$(ac - bd)(q + q^*) = \frac{(a - b)(c + d)}{p} + \frac{(c - d)(a + b)}{p},$$

and hence since $ac - bd \neq 0$, we have

$$\begin{aligned}p(q + q^*) &= \frac{(a - b)(c + d) + (c - d)(a + b)}{ac - bd} \\ &= \frac{ac + ad - bc - bd + ac + bc - ad - bd}{ac - bd} \\ &= \frac{2ac - 2bd}{ac - bd} \\ &= 2,\end{aligned}$$

as desired.

2018.3 Question 7

1. We have

$$\begin{aligned}
&\frac{(\cot\theta+i)^{2n+1}-(\cot\theta-i)^{2n+1}}{2i}\\
&=\frac{(\cos\theta+i\sin\theta)^{2n+1}-(\cos\theta-i\sin\theta)^{2n+1}}{2i\sin^{2n+1}\theta}\\
&=\frac{(\cos(2n+1)\theta+i\sin(2n+1)\theta)-(\cos(2n+1)\theta-i\sin(2n+1)\theta)}{2i\sin^{2n+1}\theta}\\
&=\frac{2i\sin(2n+1)\theta}{2i\sin^{2n+1}\theta}\\
&=\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta},
\end{aligned}$$

as desired.

By applying the binomial expansion formula on the numerator, we have

$$\begin{aligned}
&(\cot\theta+i)^{2n+1}-(\cot\theta-i)^{2n+1}\\
&=\sum_{t=0}^{2n+1}\binom{2n+1}{t}\cot^t\theta\cdot i^{2n+1-t}-\sum_{t=0}^{2n+1}\binom{2n+1}{t}\cot^t\theta\cdot(-i)^{2n+1-t}\\
&=\sum_{t=0}^{2n+1}\binom{2n+1}{t}\cot^t\theta\cdot\left[i^{2n+1-t}-(-i)^{2n+1-t}\right]\\
&=(-1)^n\cdot i\cdot\sum_{t=0}^{2n+1}\binom{2n+1}{t}\cot^t\theta\cdot i^{-t}\cdot\left[1-(-1)^{1-t}\right].
\end{aligned}$$

Due to the existence of the final term, this means that only terms with even t will retain (give a 2), and odd ts will cancel. Hence,

$$\begin{aligned}
&(\cot\theta+i)^{2n+1}-(\cot\theta-i)^{2n+1}\\
&=(-1)^n\cdot i\cdot\sum_{t=0}^{2n+1}\binom{2n+1}{t}\cot^t\theta\cdot i^{-t}\cdot\left[1-(-1)^{1-t}\right]\\
&=(-1)^n\cdot 2i\cdot\sum_{t=0}^{n}\binom{2n+1}{2t}\cot^{2t}\theta\cdot i^{-2t}\\
&=2i(-1)^n\cdot\sum_{t=0}^{n}\binom{2n+1}{2t}\cot^{2t}\theta\cdot(-1)^t\\
&=2i(-1)^n\cdot\sum_{t=0}^{n}\binom{2n+1}{2n-2t+1}\cot^{2t}\theta\cdot(-1)^t\\
&=2i(-1)^n\cdot\sum_{t=0}^{n}\binom{2n+1}{2t+1}\cot^{2(n-t)}\theta\cdot(-1)^{n-t}\\
&=2i\cdot\sum_{t=0}^{n}\binom{2n+1}{2t+1}\cot^{2(n-t)}\theta\cdot(-1)^t.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta}\\
&=\frac{2i\cdot\sum_{t=0}^{n}\binom{2n+1}{2t+1}\cot^{2(n-t)}\theta\cdot(-1)^t}{2i}\\
&=\sum_{t=0}^{n}\binom{2n+1}{2t+1}\cot^{2(n-t)}\theta\cdot(-1)^t.
\end{aligned}$$

The left-hand side of the original equation is

$$\sum_{t=0}^{n} \binom{2n+1}{2t+1} x^{n-t} \cdot (-1)^t.$$

Let $x = \cot^2 \theta$, we have

$$\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta} = \sum_{t=0}^{n} \binom{2n+1}{2t+1} x^{n-t} \cdot (-1)^t = 0.$$

Therefore, we have $\sin(2n+1)\theta = 0$, and hence $(2n+1)\theta = m\pi$ for $m \in \mathbb{Z}$.

To avoid duplicate solutions for $x = \cot^2 \theta$, we restrict $\theta \in \left(0, \frac{\pi}{2}\right]$, and hence $(2n+1)\theta \in \left(0, \left(n+\frac{1}{2}\right)\pi\right]$, and hence $m = 1, 2, \ldots, n$.

This solves to $\theta = \frac{m\pi}{2n+1}$ for $m = 1, 2, \ldots, n$, and hence this gives exactly

$$x = \cot^2\left(\frac{m\pi}{2n+1}\right).$$

2. By Vieta's Theorem, we will have

$$\sum_{m=1}^{n} x_m = -\frac{-\binom{2n+1}{3}}{\binom{2n+1}{1}} = \frac{(2n+1)(2n)(2n-1)}{(2n+1)\cdot 3 \cdot 2 \cdot 1} = \frac{n(2n-1)}{3},$$

and since we have

$$x_m = \cot^2\left(\frac{m\pi}{2n+1}\right),$$

we have

$$\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) = \frac{n(2n-1)}{3}.$$

3. For $0 < \theta < \frac{1}{2}\pi$, we have $0 < \sin\theta < \theta < \tan\theta$, and squaring this gives

$$0 < \sin^2\theta < \theta^2 < \tan^2\theta,$$

and flipping to the reciprocal gives

$$0 < \cot^2\theta < \frac{1}{\theta^2} < \csc^2\theta = 1 + \cot^2\theta,$$

which proves exactly what is desired.

Therefore, we have

$$\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) < \sum_{m=1}^{n} \frac{1}{\left(\frac{m\pi}{2n+1}\right)^2} < \sum_{m=1}^{n} \left[1 + \cot^2\left(\frac{m\pi}{2n+1}\right)\right],$$

and hence

$$\frac{n(2n-1)}{3} < \sum_{m=1}^{n} \frac{(2n+1)^2}{m^2\pi^2} < \frac{2n(n+1)}{3},$$

and hence

$$\frac{n(2n-1)\pi^2}{3(2n+1)^2} < \sum_{m=1}^{n} \frac{1}{m^2} < \frac{2n(n+1)\pi^2}{3(2n+1)^2}.$$

Take the limit as $n \to \infty$, the strict inequalities become weak, and hence

$$\lim_{n\to\infty} \frac{n(2n-1)\pi^2}{3(2n+1)^2} \leq \sum_{m=1}^{\infty} \frac{1}{m^2} \leq \lim_{n\to\infty} \frac{2n(n+1)\pi^2}{3(2n+1)^2},$$

and hence

$$\frac{2\pi^2}{3 \cdot 2^2} \leq \sum_{m=1}^{\infty} \frac{1}{m^2} \leq \frac{2n\pi^2}{3 \cdot 2^2},$$

and therefore

$$\frac{\pi^2}{6} \leq \sum_{m=1}^{\infty} \frac{1}{m^2} \leq \frac{\pi^2}{6},$$

and hence

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6},$$

as desired.

2018.3 Question 8

1. Using the substitution $t = \frac{1}{x}$, we have

$$\frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{1}{x^2} \implies \mathrm{d}x = -x^2 \,\mathrm{d}t = -\frac{\mathrm{d}t}{t^2},$$

and when $x \to 0^+$, $t \to \infty$, and when $x = 1$, $t = 1$. Hence,

$$\begin{aligned} I &= \int_0^1 \frac{f(x^{-1})}{1+x} \,\mathrm{d}x \\ &= \int_1^\infty \frac{f(t)}{1+t^{-1}} \cdot \left(-\frac{\mathrm{d}t}{t^2}\right) \\ &= \int_1^\infty \frac{f(t)\,\mathrm{d}t}{t(1+t)} \\ &= \int_1^2 \frac{f(t)\,\mathrm{d}t}{t(1+t)} + \int_2^3 \frac{f(t)\,\mathrm{d}t}{t(1+t)} + \int_3^4 \frac{f(t)\,\mathrm{d}t}{t(1+t)} + \cdots \\ &= \sum_{n=1}^\infty \int_n^{n+1} \frac{f(t)\,\mathrm{d}t}{t(1+t)}, \end{aligned}$$

as desired.

Since $f(x) = f(x+1)$ for all x, we must have that $f(x) = f(x+n)$ for all x and integers n. Also, we have

$$\frac{1}{y(1+y)} = \frac{1}{y} - \frac{1}{1+y}.$$

Hence,

$$\begin{aligned} I &= \sum_{n=1}^\infty \int_n^{n+1} \frac{f(t)\,\mathrm{d}t}{t(1+t)} \\ &= \sum_{n=1}^\infty \int_0^1 \frac{f(n+t)\,\mathrm{d}t}{(n+t)(n+t+1)} \\ &= \sum_{n=1}^\infty \int_0^1 f(t) \cdot \left[\frac{1}{n+t} - \frac{1}{n+t+1}\right] \mathrm{d}t \\ &= \sum_{n=1}^\infty \int_0^1 \frac{f(t)\,\mathrm{d}t}{n+t} - \sum_{n=1}^\infty \int_0^1 \frac{f(t)\,\mathrm{d}t}{n+t+1} \\ &= \sum_{n=1}^\infty \int_0^1 \frac{f(t)\,\mathrm{d}t}{n+t} - \sum_{n=2}^\infty \int_0^1 \frac{f(t)\,\mathrm{d}t}{n+t} \\ &= \int_0^1 \frac{f(t)\,\mathrm{d}t}{1+t}. \end{aligned}$$

2. For the first integral, simply consider $f(x) = \{x\}$, and we can immediately see that $f(x)$ has period of 1 from the definition. Hence,

$$\int_0^1 \frac{\{x^{-1}\}}{1+x} \,\mathrm{d}x = \int_0^1 \frac{f(x^{-1})}{1+x} \,\mathrm{d}x = \int_0^1 \frac{f(x)}{1+x} \,\mathrm{d}x = \int_0^1 \frac{\{x\}}{1+x} \,\mathrm{d}x.$$

Since for $0 < x < 1$, we have $\{x\} = x$, and hence

$$\begin{aligned}
\int_0^1 \frac{\{x^{-1}\}}{1+x} \, \mathrm{d}x &= \int_0^1 \frac{\{x\}}{1+x} \, \mathrm{d}x \\
&= \int_0^1 \frac{x}{1+x} \, \mathrm{d}x \\
&= \int_0^1 \left(1 - \frac{1}{1+x}\right) \mathrm{d}x \\
&= 1 - [\ln(1+x)]_0^1 \\
&= 1 - (\ln(2) - \ln(1)) \\
&= 1 - \ln 2.
\end{aligned}$$

For the second integral, we let $g(x) = \{2x\}$, and we can see that $g(x)$ has a period of $\frac{1}{2}$, and hence it also has a period of 1. Hence,

$$\int_0^1 \frac{\{2x^{-1}\}}{1+x} \, \mathrm{d}x = \int_0^1 \frac{g(x^{-1})}{1+x} \, \mathrm{d}x = \int_0^1 \frac{g(x)}{1+x} \, \mathrm{d}x = \int_0^1 \frac{\{2x\}}{1+x} \, \mathrm{d}x.$$

We split this integral into two parts, $[0, 0.5]$ and $[0.5, 1]$.

$$\begin{aligned}
\int_0^1 \frac{\{2x^{-1}\}}{1+x} \, \mathrm{d}x &= \int_0^1 \frac{\{2x\}}{1+x} \, \mathrm{d}x \\
&= \int_0^{0.5} \frac{\{2x\}}{1+x} \, \mathrm{d}x + \int_{0.5}^1 \frac{\{2x\}}{1+x} \, \mathrm{d}x \\
&= \int_0^{0.5} \frac{2x}{1+x} \, \mathrm{d}x + \int_{0.5}^1 \frac{2x-1}{1+x} \, \mathrm{d}x \\
&= \int_0^{0.5} \left[2 - \frac{2}{1+x}\right] \mathrm{d}x + \int_{0.5}^1 \left[2 - \frac{3}{1+x}\right] \mathrm{d}x \\
&= 1 - 2\left[\ln(1+x)\right]_0^{0.5} + 1 - 3\left[\ln(1+x)\right]_{0.5}^1 \\
&= 2 - 2\ln 1.5 + 2\ln 1 - 3\ln 2 + 3\ln 1.5 \\
&= 2 - 3\ln 2 + \ln 1.5 \\
&= 2 - 3\ln 2 + \ln 3 - \ln 2 \\
&= 2 - 4\ln 2 + \ln 3.
\end{aligned}$$

2018.3 Question 12

1. $\mathrm{P}(Y_k) \leq y$ is the probability that there is at least k numbers that are less than equal to y.

   If there are $k \leq m \leq n$ numbers less than or equal to y, then there must be $n-m$ numbers greater than or equal to y. The probability of the first thing happening for each number is y, and for the second thing happening for each number is $1-y$. We also have to choose m numbers from the n to make them less than or equal to y. Therefore,

$$\mathrm{P}(Y_k \leq y) = \sum_{m=k}^{n} \binom{n}{m} y^m (1-y)^{n-m}.$$

2. We have

$$m\binom{n}{m} = m \cdot \frac{n!}{m!(n-m)!} = \frac{n!}{(m-1)!(n-m)!} = n \cdot \frac{(n-1)!}{(m-1)!(n-m)!} = n\binom{n-1}{m-1}.$$

   We have

$$(n-m)\binom{n}{m} = (n-m) \cdot \frac{n!}{m!(n-m)!} = \frac{n!}{m!(n-m-1)!} = n \cdot \frac{(n-1)!}{m!(n-m-1)!} = n\binom{n-1}{m}.$$

   The cumulative distribution function F_{Y_k} is

$$F_{Y_k}(y) = \sum_{m=k}^{n} \binom{n}{m} y^m (1-y)^{n-m}.$$

   Therefore, the probability density function f_{Y_k} is

$$\begin{aligned}
f_{Y_k}(y) &= F'_{Y_k}(y) \\
&= \sum_{m=k}^{n} \binom{n}{m} \left[m y^{m-1}(1-y)^{n-m} - (n-m) y^m (1-y)^{n-m-1} \right] \\
&= \sum_{m=k}^{n} y^{m-1}(1-y)^{n-m-1} \left[m\binom{n}{m}(1-y) - (n-m)\binom{n}{m} y \right] \\
&= n \left[\sum_{m=k}^{n} \binom{n-1}{m-1} y^{m-1}(1-y)^{n-m} - \sum_{m=k}^{n-1} \binom{n-1}{m} y^m (1-y)^{n-m-1} \right] \\
&= n \left[\sum_{m=k}^{n} \binom{n-1}{m-1} y^{m-1}(1-y)^{n-m} - \sum_{m=k+1}^{n} \binom{n-1}{m-1} y^{m-1}(1-y)^{n-m} \right] \\
&= n\binom{n-1}{k-1} y^{k-1}(1-y)^{n-k}.
\end{aligned}$$

   Since $Y_k \in [0, 1]$, we must have

$$\int_0^1 f_{Y_k}(y) \, \mathrm{d}y = 1,$$

   and hence

$$n\binom{n-1}{k-1} \int_0^1 y^{k-1}(1-y)^{n-k} \, \mathrm{d}y = 1,$$

   and therefore we have

$$\int_0^1 y^{k-1}(1-y)^{n-k} \, \mathrm{d}y = \frac{1}{n\binom{n-1}{k-1}}.$$

3. By the definition of the expectation,

$$\begin{aligned}
\mathrm{E}(Y_k) &= \int_0^1 y f_{Y_k}(y)\,\mathrm{d}y \\
&= n\binom{n-1}{k-1}\int_0^1 y^k(1-y)^{n-k}\,\mathrm{d}y \\
&= n\binom{n-1}{k-1}\cdot\frac{1}{(n+1)\binom{n}{k}} \\
&= \frac{n\cdot\frac{(n-1)!}{(k-1)!(n-k)!}}{(n+1)\cdot\frac{n!}{k!(n-k)!}} \\
&= \frac{\frac{n!}{(k-1)!(n-k)!}}{\frac{(n+1)n!}{k(k-1)!(n-k)!}} \\
&= \frac{k}{n+1}.
\end{aligned}$$

2018.3 Question 13

By the definition of a probability generating function, we have

$$G(1)=\sum_{n=0}^{\infty}\mathrm{P}(X=n),\ \text{and}\ G(-1)=\sum_{n=0}^{\infty}(-1)^n\,\mathrm{P}(X=n).$$

Hence,

$$G(1)+G(-1)=\sum_{n=0}^{\infty}[1+(-1)^n]\,\mathrm{P}(X=n).$$

When n is odd, $1+(-1)^n=0$. When n is even, $1+(-1)^n=2$.
This means

$$G(1)+G(-1)=2\sum_{n=0}^{\infty}\mathrm{P}(X=2n),$$

which gives

$$\frac{1}{2}(G(1)+G(-1))=\sum_{n=0}^{\infty}\mathrm{P}(X=2n)=\mathrm{P}(X=0\text{ or }X=2\text{ or }X=4\ldots).$$

Since $X\sim\mathrm{Po}(\lambda)$, we have

$$\mathrm{P}(X=x)=e^{-\lambda}\frac{\lambda^x}{x!},$$

and hence the probability generating function for X, $G(t)$, must satisfy

$$\begin{aligned}G(t)&=\sum_{n=0}^{\infty}\mathrm{P}(X=n)\cdot t^n\\&=\sum_{n=0}^{\infty}e^{-\lambda}\frac{\lambda^n}{n!}\cdot t^n\\&=e^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}\\&=e^{-\lambda}\cdot e^{\lambda t}\\&=e^{-\lambda(1-t)}.\end{aligned}$$

1. Consider $G(t)+G(-t)$. By definition, we have

$$G(t)=\sum_{n=0}^{\infty}\mathrm{P}(X=n)t^n, G(-t)=\sum_{n=0}^{\infty}(-1)^n\,\mathrm{P}(X=n)t^n,$$

and hence

$$G(t)+G(-t)=\sum_{n=0}^{\infty}(1+(-1)^n)\,\mathrm{P}(X=n)t^n=2\sum_{n=0}^{\infty}\mathrm{P}(X=2n)t^{2n}.$$

Let $H(t)$ be the probability generating function of Y, we have

$$\begin{aligned}H(t)&=\sum_{n=0}^{\infty}\mathrm{P}(Y=n)\cdot t^n\\&=\sum_{n=0}^{\infty}\mathrm{P}(Y=2n)\cdot t^{2n}\\&=\sum_{n=0}^{\infty}k\,\mathrm{P}(X=2n)\cdot t^{2n}\\&=\frac{k}{2}\left(G(t)+G(-t)\right).\end{aligned}$$

To find k, we must have $H(1) = 1$. Hence,

$$1 = \frac{k}{2}\left(G(1) + G(-1)\right) = \frac{k}{2}\left(e^{-\lambda(1-1)} + e^{-\lambda(1+1)}\right) = \frac{k}{2}\left(1 + e^{-2\lambda}\right),$$

which gives

$$k = \frac{2}{1 + e^{-2\lambda}} = \frac{2e^{\lambda}}{e^{\lambda} + e^{-\lambda}} = \frac{e^{\lambda}}{\cosh \lambda}.$$

Hence,

$$\begin{aligned} H(t) &= \frac{k}{2}\left(G(t) + G(-t)\right) \\ &= \frac{e^{\lambda}}{2\cosh \lambda}\left(e^{-\lambda(1-t)} + e^{-\lambda(1+t)}\right) \\ &= \frac{1}{\cosh \lambda}\frac{e^{\lambda t} + e^{-\lambda t}}{2} \\ &= \frac{\cosh \lambda t}{\cosh \lambda}. \end{aligned}$$

Differentiating this with respect to t, we have

$$H'(t) = \frac{\lambda \sinh \lambda t}{\cosh \lambda},$$

and hence

$$\mathrm{E}(Y) = H'(1) = \frac{\lambda \sinh \lambda \cdot 1}{\cosh \lambda} = \lambda \tanh \lambda.$$

Since $-1 < \tanh \lambda < 1$, we have $\lambda \tanh \lambda < \lambda$, and so $\mathrm{E}(Y) < \lambda$ for $\lambda > 0$.

2. Consider $G(t) + G(-t) + G(it) + G(-it)$. By definition, we have

$$G(t) + G(-t) + G(it) + G(-it) = \sum_{n=0}^{\infty} \left(1 + (-1)^n + i^n + (-i)^n\right) \mathrm{P}(X = n) \cdot t^n.$$

Let m be an integer. Consider the following four cases:

- $n = 4m$, $1 + (-1)^n + i^n + (-i)^n = 1 + 1 + 1 + 1 = 4$.
- $n = 4m + 1$, $1 + (-1)^n + i^n + (-i)^n = 1 + (-1) + i + (-i) = 0$.
- $n = 4m + 2$, $1 + (-1)^n + i^n + (-i)^n = 1 + 1 + (-1) + (-1) = 0$.
- $n = 4m + 3$, $1 + (-1)^n + i^n + (-i)^n + 1 + (-1) + (-i) + i = 0$.

Hence,

$$G(t) + G(-t) + G(it) + G(-it) = 4\sum_{n=0}^{\infty} \mathrm{P}(X = 4n) \cdot t^{4n}.$$

Let $P(t)$ be the probability generating function of Z, we have

$$\begin{aligned} P(t) &= \sum_{n=0}^{\infty} \mathrm{P}(Z = n) \cdot t^n \\ &= \sum_{n=0}^{\infty} \mathrm{P}(Z = 4n) \cdot t^{4n} \\ &= c\sum_{n=0}^{\infty} \mathrm{P}(X = 4n) \cdot t^{4n} \\ &= \frac{c}{4}\left(G(t) + G(-t) + G(it) + G(-it)\right). \end{aligned}$$

Since $P(1) = 0$, we must have

$$\begin{aligned}
1 &= \frac{c}{4}\left(G(1) + G(-1) + G(i) + G(-i)\right) \\
&= \frac{c}{4}\left(e^{-\lambda(1-1)} + e^{-\lambda(1+1)} + e^{-\lambda(1-i)} + e^{-\lambda(1+i)}\right) \\
&= \frac{ce^{-\lambda}}{4}\left(e^{\lambda} + e^{-\lambda} + e^{i\lambda} + e^{-i\lambda}\right) \\
&= \frac{ce^{-\lambda}}{2}\left(\cos\lambda + \cosh\lambda\right).
\end{aligned}$$

Hence,

$$c = \frac{2e^{\lambda}}{\cos\lambda + \cosh\lambda}.$$

Therefore,

$$\begin{aligned}
P(t) &= \frac{c}{4}\left(G(t) + G(-t) + G(it) + G(-it)\right) \\
&= \frac{e^{\lambda}}{2(\cos\lambda + \cosh\lambda)}\left[e^{-\lambda(1-t)} + e^{-\lambda(1+t)} + e^{-\lambda(1-it)} + e^{-\lambda(1+it)}\right] \\
&= \frac{e^{\lambda t} + e^{-\lambda t} + e^{\lambda it} + e^{-\lambda it}}{2(\cos\lambda + \cosh\lambda)} \\
&= \frac{\cos\lambda t + \cosh\lambda t}{\cos\lambda + \cosh\lambda}.
\end{aligned}$$

Differentiating this with respect to t gives us

$$P'(t) = \frac{\lambda(-\sin\lambda t + \sinh\lambda t)}{\cos\lambda + \cosh\lambda},$$

and hence

$$\mathrm{E}(Z) = P'(1) = \frac{\lambda(-\sin\lambda + \sinh\lambda)}{\cos\lambda + \cosh\lambda}.$$

$\mathrm{E}(Z) < \lambda$ is equivalent to

$$\frac{\sinh\lambda - \sin\lambda}{\cosh\lambda + \cos\lambda} < 1,$$

which is then equivalent to

$$\sinh\lambda - \cosh\lambda < \sin\lambda + \cos\lambda,$$

which is

$$-e^{-\lambda} < \sin\lambda + \cos\lambda.$$

However, this is not necessarily true. Let $\lambda = \pi$. We have

$$\text{LHS} = -e^{-\pi} > -e^{0} = -1,$$

and

$$\text{RHS} = \sin\pi + \cos\pi = -1,$$

which means LHS > RHS for $\lambda = \pi$, which means $\mathrm{E}(Z) > \lambda$. Therefore, the statement is not true.