Year 2017

2017 Paper 3

2017.3 Question 1

1. We have

$$\begin{aligned}
\text{RHS} &= \frac{r+1}{r}\left(\frac{1}{\binom{n+r-1}{r}} - \frac{1}{\binom{n+r}{r}}\right) \\
&= \frac{r+1}{r}\left(\frac{r!(n-1)!}{(n+r-1)!} - \frac{r!n!}{(n+r)!}\right) \\
&= \frac{r+1}{r}\left(\frac{r!(n-1)!(n+r)}{(n+r)!} - \frac{r!(n-1)!n}{(n+r)!}\right) \\
&= \frac{r+1}{r} \cdot \frac{r!(n-1)!(n+r) - r!(n-1)!n}{(n+r)!} \\
&= \frac{r+1}{r} \cdot \frac{r!(n-1)!r}{(n+r)!} \\
&= \frac{(r+1)!(n-1)!}{(n+r)!} \\
&= \binom{n+r}{r+1} \\
&= \text{LHS}
\end{aligned}$$

as desired.

Therefore,

$$\begin{aligned}
\sum_{n=1}^{+\infty} \frac{1}{\binom{n+r}{r+1}} &= \sum_{n=1}^{+\infty} \frac{r+1}{r}\left(\frac{1}{\binom{n+r-1}{r}} - \frac{1}{\binom{n+r}{r}}\right) \\
&= \frac{r+1}{r}\sum_{n=1}^{+\infty}\left(\frac{1}{\binom{n+r-1}{r}} - \frac{1}{\binom{n+r}{r}}\right) \\
&= \frac{r+1}{r}\left[\sum_{n=0}^{+\infty}\frac{1}{\binom{n+r}{r}} - \sum_{n=1}^{+\infty}\frac{1}{\binom{n+r}{r}}\right] \\
&= \frac{r+1}{r}\frac{1}{\binom{0+r}{r}} \\
&= \frac{r+1}{r},
\end{aligned}$$

assuming the sum converges.

When $r = 2$, we have

$$\sum_{n=1}^{+\infty} \frac{1}{\binom{n+2}{3}} = \frac{3}{2}.$$

When $n = 1$, $\frac{1}{\binom{1+2}{3}} = \frac{1}{1} = 1$.

Therefore,

$$\sum_{n=2}^{+\infty} \frac{1}{\binom{n+2}{3}} = \frac{1}{2}$$

as desired.

2. Notice that

$$\begin{aligned}
\frac{3!}{n^3} < \frac{1}{\binom{n+1}{3}} &\iff \frac{3!}{n^3} < \frac{3!}{(n+1)n(n-1)} \\
&\iff n^3 > (n+1)n(n-1) \\
&\iff n^3 > n(n^2-1) \\
&\iff n^3 > n^3 - n \\
&\iff n > 0,
\end{aligned}$$

which is true.

Also, notice that

$$\begin{aligned}
\frac{20}{\binom{n+1}{3}} - \frac{1}{\binom{n+2}{5}} < \frac{5!}{n^3} &\iff \frac{5!}{(n+1)(n)(n-1)} - \frac{5!}{(n+2)(n+1)(n)(n-1)(n-2)} < \frac{5!}{n^3} \\
&\iff \frac{(n+2)(n-2)-1}{(n+2)(n+1)(n)(n-1)(n-2)} < \frac{1}{n^3} \\
&\iff (n^2-5)n^3 < (n^2-4)(n^2-1)n \\
&\iff n^5 - 5n^3 < n^5 - 5n^3 + 4n \\
&\iff 4n > 0,
\end{aligned}$$

which is true.

Therefore, we have that

$$\begin{aligned}
\sum_{n=3}^{+\infty} \frac{3!}{n^3} &< \sum_{n=3}^{+\infty} \frac{1}{\binom{n+1}{3}} \\
&= \sum_{n=2}^{+\infty} \frac{1}{\binom{n+2}{3}} \\
&= \frac{1}{2},
\end{aligned}$$

and therefore $\sum_{n=3}^{+\infty} \frac{1}{n^3} < \frac{1}{12}$, and $\sum_{n=1}^{+\infty} \frac{1}{n^3} < 1 + \frac{1}{8} + \frac{1}{12} = \frac{29}{24} = \frac{116}{96}$.

On the other hand, we have

$$\begin{aligned}
\sum_{n=3}^{+\infty} \frac{5!}{n^3} &< \sum_{n=3}^{+\infty} \left[\frac{20}{\binom{n+1}{3}} - \frac{1}{\binom{n+2}{5}} \right] \\
&= 20 \sum_{n=2}^{+\infty} \frac{1}{\binom{n+2}{3}} - \sum_{n=1}^{+\infty} \frac{1}{\binom{n+4}{5}} \\
&= 20 \cdot \frac{1}{2} - \frac{5}{4} \\
&= 10 - \frac{5}{4} \\
&= \frac{35}{4},
\end{aligned}$$

and therefore $\sum_{n=3}^{+\infty} \frac{1}{n^3} > \frac{7}{96}$, and $\sum_{n=1}^{+\infty} \frac{1}{n^3} > 1 + \frac{1}{8} + \frac{7}{96} = \frac{115}{96}$.

Hence,

$$\frac{115}{96} < \sum_{n=1}^{+\infty} \frac{1}{n^3} < \frac{116}{96}$$

as desired.

2017.3 Question 2

1. Let the complex number representing $R(P)$ be z'. Therefore,

$$\begin{aligned} z' - a &= \exp(i\theta)(z-a), \\ z' &= z\exp(i\theta) + a(1-\exp(i\theta)), \end{aligned}$$

as desired.

2. Let the complex number representing $SR(P)$ be z''. Therefore,

$$\begin{aligned} z'' - b &= \exp(i\varphi)(z'-b), \\ z'' &= z'\exp(i\varphi) + b(1-\exp(i\varphi)), \\ z'' &= [z\exp(i\theta) + a(1-\exp(i\theta))]\exp(i\varphi) + b(1-\exp(i\varphi)), \\ z'' &= z\exp(i\,(\theta+\varphi)) + a(1-\exp(i\theta))\exp(i\varphi) + b(1-\exp(i\varphi)). \end{aligned}$$

This will be an anti-clockwise rotation around c over an angle of $(\theta+\varphi)$, where

$$c\,[1-\exp(i(\theta+\varphi))] = a\exp(i\varphi) - a\exp(i\,(\theta+\varphi)) + b - b\exp(i\varphi),$$

If $\theta+\varphi = 2n\pi$ for some integer $n \in \mathbb{Z}$, $1-\exp(i(\theta+\varphi)) = 0$, therefore c cannot be determined. Multiplying both sides by $\exp\left(-\frac{i(\theta+\varphi)}{2}\right)$, we have

$$\begin{aligned} & c\left[\exp\left(-\frac{i(\theta+\varphi)}{2}\right) - \exp\left(\frac{i(\theta+\varphi)}{2}\right)\right] \\ &= a\left[\exp\left(\frac{i(\varphi-\theta)}{2}\right) - \exp\left(\frac{i(\theta+\varphi)}{2}\right)\right] + b\left[\exp\left(-\frac{i(\theta+\varphi)}{2}\right) - \exp\left(\frac{i(\varphi-\theta)}{2}\right)\right], \end{aligned}$$

and hence

$$\begin{aligned} -2ci\sin\left(\frac{\theta+\varphi}{2}\right) &= -2ai\exp\left(\frac{i\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right) - 2bi\exp\left(-\frac{i\theta}{2}\right)\sin\left(\frac{\varphi}{2}\right), \\ c\sin\left(\frac{\theta+\varphi}{2}\right) &= a\exp\left(\frac{i\varphi}{2}\right)\sin\left(\frac{\theta}{2}\right) + b\exp\left(-\frac{i\theta}{2}\right)\sin\left(\frac{\varphi}{2}\right). \end{aligned}$$

If $\theta+\varphi = 2\pi$, we will have $z'' = z + a\exp(i\varphi) - a + b(1-\exp(i\varphi)) = z + (b-a)(1-\exp(i\varphi))$, which is a translation by $(b-a)(1-\exp(i\varphi))$.

3. If $RS = SR$, then we have

$$\begin{aligned} a(1-\exp(i\theta))\exp(i\varphi) + b(1-\exp(i\varphi)) &= b(1-\exp(i\varphi))\exp(i\theta) + a(1-\exp(i\theta)), \\ a(-1+\exp(i\varphi)+\exp(i\theta)-\exp(i(\theta+\varphi))) &= b(-1+\exp(i\varphi)+\exp(i\theta)-\exp(i(\theta+\varphi))), \\ (a-b)(1-\exp(i\varphi))(1-\exp(i\theta)) &= 0. \end{aligned}$$

Therefore, $a = b$, or $\varphi = 2n\pi$, or $\theta = 2n\pi$, for some integer $n \in \mathbb{Z}$.

2017.3 Question 3

By Vieta's Theorem, from the quartic equation in x, we have

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q,$$

and from the cubic equation in y, we have

$$(\alpha\beta + \gamma\delta) + (\alpha\gamma + \beta\delta) + (\alpha\delta + \beta\gamma) = -A.$$

Therefore, $A = -q$.

1. Since $(p, q, r, s) = (0, 3, -6, 10)$, the cubic equation is reduced to

   $$y^3 - 3y^2 - 10y + 84 = 0,$$

   and therefore

   $$(y - 2)(y - 7)(y + 6) = 0.$$

   Therefore, $y_1 = 7, y_2 = 2, y_3 = -6$, and $\alpha\beta + \gamma\delta = 7$.

2. We have

   $$\begin{aligned}(\alpha + \beta)(\gamma + \delta) &= \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta \\ &= (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) - (\alpha\beta + \gamma\delta) \\ &= q - 7 \\ &= 3 - 7 \\ &= -4.\end{aligned}$$

   By Vieta's Theorem, we have $\alpha\beta\gamma\delta = s = 10$. Therefore, $\alpha\beta$ and $\gamma\delta$ must be roots to the equation

   $$x^2 - 7x + 10 = 0.$$

   The two roots are $x = 2$ and $x = 5$, and therefore $\alpha\beta = 5$.

3. We have from the other root that $\gamma\delta = 2$.

   We notice that $(\alpha + \beta) + (\gamma + \delta) = -p = 0$. Therefore, from part 2, $(\alpha + \beta)$ and $(\gamma + \delta)$ are roots to the equation

   $$x^2 - 4 = 0.$$

   This gives us $\alpha + \beta = \pm 2$ and $\gamma + \delta = \mp 2$.

   Using the value of r and Vieta's Theorem, we have

   $$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r = 6.$$

   Plugging in $\alpha\beta = 5$ and $\gamma\delta = 2$, we have

   $$5(\gamma + \delta) + 2(\alpha + \beta) = 6.$$

   Therefore, it must be the case that $\alpha + \beta = -2$ and $\gamma + \delta = 2$.

   Hence, using the values of $\alpha\beta$ and $\gamma\delta$, α and β are solutions to the quadratic equation $x^2+2x+5 = 0$, and γ and δ are solutions to the quadratic equation $x^2 - 2x + 2 = 0$.

   Solving this gives us $\alpha, \beta = -1 \pm 2i$ and $\gamma, \delta = 1 \pm i$. The solutions to the original quartic equation is

   $$x_{1,2} = -1 \pm 2i, x_{3,4} = 1 \pm i.$$

2017.3 Question 4

1. Notice that $a = e^{\ln a}$ and hence $a^x = e^{x \ln a}$, $a^{\frac{x}{\ln a}} = e^x$we have

$$\begin{aligned} F(y) &= \exp\left(\frac{1}{y}\int_0^y \ln f(x)\,\mathrm{d}x\right) \\ &= a^{\frac{1}{y\ln a}\cdot\int_0^y \ln f(x)\,\mathrm{d}x} \\ &= a^{\frac{1}{y}\cdot\int_0^y \frac{\ln f(x)}{\ln a}\,\mathrm{d}x} \\ &= a^{\frac{1}{y}\cdot\int_0^y \log_a f(x)\,\mathrm{d}x} \end{aligned}$$

as desired.

2. We have

$$\begin{aligned} H(y) &= \exp\left(\frac{1}{y}\int_0^y \ln f(x)g(x)\,\mathrm{d}x\right) \\ &= \exp\left[\frac{1}{y}\int_0^y (\ln f(x) + \ln g(x))\,\mathrm{d}x\right] \\ &= \exp\left[\frac{1}{y}\left(\int_0^y \ln f(x)\,\mathrm{d}x + \int_0^y \ln g(x)\,\mathrm{d}x\right)\right] \\ &= \exp\left(\frac{1}{y}\int_0^y \ln f(x)\,\mathrm{d}x\right)\cdot\exp\left(\frac{1}{y}\int_0^y \ln g(x)\,\mathrm{d}x\right) \\ &= F(y)\cdot G(y). \end{aligned}$$

3. Let $f(x) = b^x$.

$$\begin{aligned} F(y) &= \exp\left(\frac{1}{y}\int_0^y \ln f(x)\,\mathrm{d}x\right) \\ &= b^{\frac{1}{y}\int_0^y \log_b f(x)\,\mathrm{d}x} \\ &= b^{\frac{1}{y}\int_0^y \log_b b^x\,\mathrm{d}x} \\ &= b^{\frac{1}{y}\int_0^y x\,\mathrm{d}x} \\ &= b^{\frac{1}{y}\cdot\frac{y^2}{2}} \\ &= b^{\frac{y}{2}} \\ &= \sqrt{b^y}. \end{aligned}$$

4. Since $F(y) = \sqrt{f(y)}$, we notice that $f(y) = F(y)^2 = \exp\left(\frac{2}{y}\int_0^y \ln f(x)\,\mathrm{d}x\right)$, and therefore $\ln f(y) = \frac{2}{y}\int_0^y \ln f(x)\,\mathrm{d}x$.

We substitute $g(y) = \ln f(y)$, and therefore

$$yg(y) = 2\int_0^y g(y)\,\mathrm{d}x.$$

Therefore, differentiating both sides with respect to y gives us

$$yg'(y) + g(y) = 2g(y),$$

and therefore

$$-g(y) + yg'(y) = 0.$$

Multiplying y^{-2} on both sides gives us

$$-y^{-2}g(y) + y^{-1}g'(y) = 0,$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}y}\frac{g(y)}{y} = 0,$$

and therefore

$$\frac{g(y)}{y} = C \implies g(y) = Cy.$$

Therefore, we have

$$\begin{aligned} f(y) &= \exp g(y) \\ &= \exp(Cy) \\ &= b^y \end{aligned}$$

if we substitute $b = \exp(C) > 0$, and therefore $f(x) = b^y$ as desired.

2017.3 Question 5

Since we have $x = r\cos\theta$ and $y = r\sin\theta$, and $r = f(\theta)$, we have

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}\theta} &= \frac{\mathrm{d}r}{\mathrm{d}\theta}\cdot\cos\theta + r\cdot\frac{\mathrm{d}\cos\theta}{\mathrm{d}\theta}\\ &= f'(\theta)\cos\theta - f(\theta)\sin\theta,\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}\theta} &= \frac{\mathrm{d}r}{\mathrm{d}\theta}\cdot\sin\theta + r\cdot\frac{\mathrm{d}\sin\theta}{\mathrm{d}\theta}\\ &= f'(\theta)\sin\theta + f(\theta)\cos\theta,\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}}\\ &= \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}\\ &= \frac{f'(\theta)\tan\theta + f(\theta)}{f'(\theta) - f(\theta)\tan\theta}.\end{aligned}$$

For the two curves, we must have

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_f \cdot \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_g = -1$$

for them to meet at right angles. Therefore,

$$\begin{aligned}\frac{f'(\theta)\tan\theta + f(\theta)}{f'(\theta) - f(\theta)\tan\theta}\cdot\frac{g'(\theta)\tan\theta + g(\theta)}{g'(\theta) - g(\theta)\tan\theta} &= -1\\ (f'(\theta)\tan\theta + f(\theta))\cdot(g'(\theta)\tan\theta + g(\theta)) &= -(f'(\theta) - f(\theta)\tan\theta)\cdot(g'(\theta) - g(\theta)\tan\theta)\\ f'(\theta)g'(\theta)(1+\tan^2\theta) + f(\theta)g(\theta)(1+\tan^2\theta) &= 0\\ f'(\theta)g'(\theta) + f(\theta)g(\theta) &= 0.\end{aligned}$$

We have $f\left(-\frac{\pi}{2}\right) = 4$. Let

$$g_a(\theta) = a(1+\sin\theta).$$

Therefore,

$$g_a'(\theta) = a\cos\theta,$$

and we have

$$f'(\theta)(a\cos\theta) + f(\theta)a(1+\sin\theta) = 0,$$

and therefore

$$\frac{\mathrm{d}f(\theta)}{\mathrm{d}\theta}\cos\theta = -f(\theta)(1+\sin\theta).$$

By separating variables we have

$$\frac{\mathrm{d}f(\theta)}{f(\theta)} = -\frac{\mathrm{d}\theta(1+\sin\theta)}{\cos\theta}.$$

Notice that

$$-\frac{1+\sin\theta}{\cos\theta} = -\frac{(1-\sin\theta)(1+\sin\theta)}{(1-\sin\theta)\cos\theta} = -\frac{\cos\theta}{1-\sin\theta} = \frac{\cos\theta}{\sin\theta - 1},$$

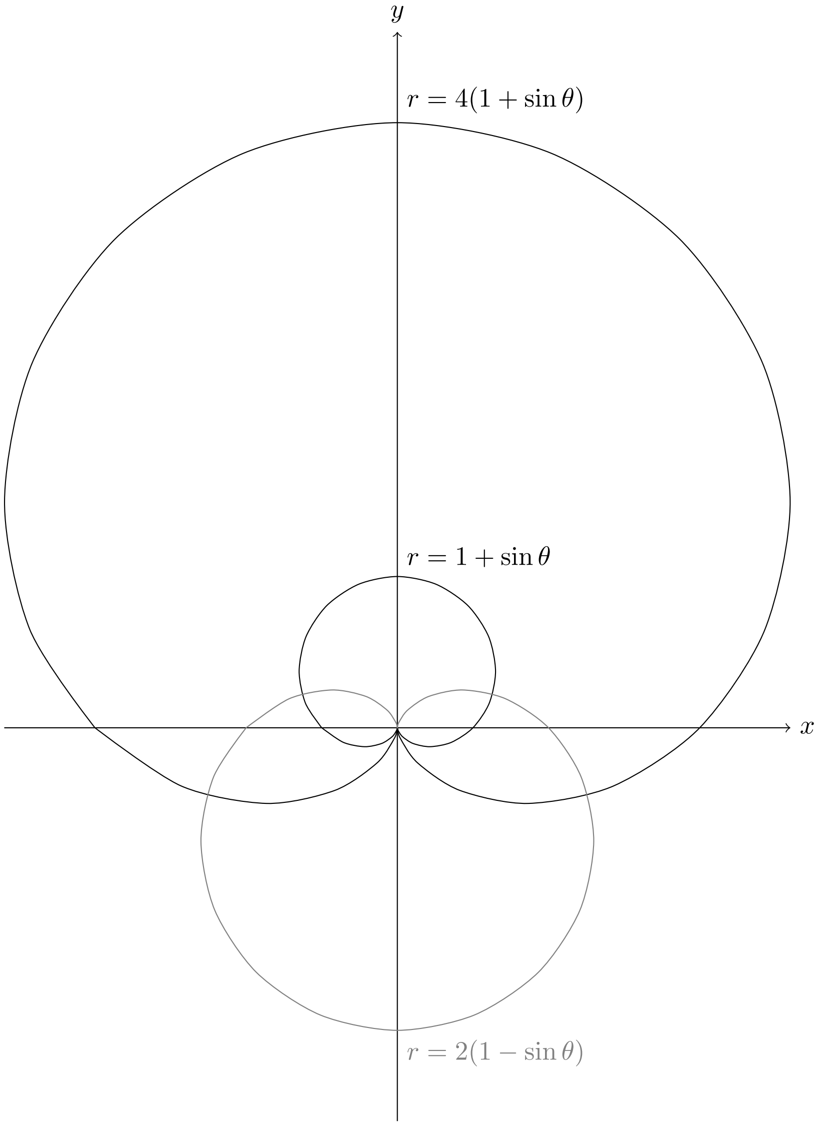
integrating both sides gives us

$$\ln f(\theta) = \ln|\sin\theta - 1| + C = \ln(1-\sin\theta) + C,$$

which gives

$$f(\theta) = A(1-\sin\theta).$$

Since $f\left(-\frac{\pi}{2}\right) = 4$, we have $2A = 4$ and $A = 2$, therefore $f(\theta) = 2(1-\sin\theta)$.


y
$r = 4(1 + \sin\theta)$
$r = 1 + \sin\theta$
x
$r = 2(1 - \sin\theta)$

2017.3 Question 6

1. Consider the substitution $u = \frac{1}{v}$.

   When $u \to 0^+$, $v \to +\infty$.

   When $u = x$, $v = \frac{1}{x}$.

   We also have

   $$\mathrm{d}u = -\frac{1}{v^2}\,\mathrm{d}v.$$

   Therefore,

   $$\begin{aligned} T(x) &= \int_0^x \frac{\mathrm{d}u}{1+u^2} \\ &= \int_{+\infty}^{\frac{1}{x}} -\frac{1}{v^2} \cdot \frac{1}{1+\frac{1}{v^2}}\,\mathrm{d}v \\ &= \int_{\frac{1}{x}}^{+\infty} \frac{\mathrm{d}v}{1+v^2} \\ &= \int_0^{+\infty} \frac{\mathrm{d}v}{1+v^2} - \int_0^{\frac{1}{x}} \frac{\mathrm{d}v}{1+v^2} \\ &= T_\infty - T(x^{-1}), \end{aligned}$$

   as desired.

2. When $u \neq a^{-1}$, we have

   $$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}u} &= \frac{\mathrm{d}}{\mathrm{d}u}\frac{u+a}{1-au} \\ &= \frac{1\cdot(1-au) + a\cdot(u+a)}{(1-au)^2} \\ &= \frac{1-au+au+a^2}{(1-au)^2} \\ &= \frac{1+a^2}{(1-au)^2}. \end{aligned}$$

   Also, notice that

   $$\begin{aligned} \frac{1+v^2}{1+u^2} &= \frac{1+\left(\frac{u+a}{1-au}\right)^2}{1+u^2} \\ &= \frac{(1-au)^2+(u+a)^2}{(1+u^2)(1-au)^2} \\ &= \frac{1-2au+a^2u^2+u^2+2au+a^2}{(1+u^2)(1-au)^2} \\ &= \frac{(1+a^2)(1+u^2)}{(1-au)^2(1+u^2)} \\ &= \frac{1+a^2}{(1-au)^2}. \end{aligned}$$

   Therefore, $\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{1+v^2}{1+u^2}$ as desired.

Consider the substitution $v = \frac{u+a}{1-au}$. When $u = 0$, $v = a$. When $u = x, v = \frac{x+a}{1-ax}$. Therefore,

$$\begin{aligned} T(x) &= \int_0^x \frac{\mathrm{d}u}{1+u^2} \\ &= \int_a^{\frac{x+a}{1-ax}} \frac{1+u^2}{1+v^2} \cdot \frac{\mathrm{d}v}{1+u^2} \\ &= \int_a^{\frac{x+a}{1-ax}} \frac{\mathrm{d}v}{1+v^2} \\ &= \int_0^{\frac{x+a}{1-ax}} \frac{\mathrm{d}v}{1+v^2} - \int_0^a \frac{\mathrm{d}v}{1+v^2} \\ &= T\left(\frac{x+a}{1-ax}\right) - T(a), \end{aligned}$$

as desired.

If we substitute $T(x) = T_\infty - T(x^{-1})$ and $T(a) = T_\infty - T(a^{-1})$, we can see that

$$\begin{aligned} T(x) &= T\left(\frac{x+a}{1-ax}\right) - T(a) \\ T_\infty - T(x^{-1}) &= T\left(\frac{x+a}{1-ax}\right) - \left[T_\infty - T(a^{-1})\right] \\ T(x^{-1}) &= 2T_\infty - T\left(\frac{x+a}{1-ax}\right) - T(a^{-1}), \end{aligned}$$

as desired.

Now, let $y = x^{-1}$ and $b = a^{-1}$. Then

$$\begin{aligned} \frac{x+a}{1-ax} &= \frac{y^{-1}+b^{-1}}{1-b^{-1}y^{-1}} \\ &= \frac{b+y}{by-1}. \end{aligned}$$

This gives us

$$T(y) = 2T_\infty - T\left(\frac{b+y}{by-1}\right) - T(b),$$

as desired.

3. Let $y = b = \sqrt{3}$. We can easily verify that $b > 0$ and $y > \frac{1}{b}$. Therefore,

$$T(\sqrt{3}) = 2T_\infty - T\left(\frac{\sqrt{3}+\sqrt{3}}{3-1}\right) - T(\sqrt{3}),$$

which simplified, gives us $T(\sqrt{3}) = \frac{2}{3}T_\infty$ as desired.

In $T(x) = T\left(\frac{x+a}{1-ax}\right) - T(a)$, let $x = a = \sqrt{2} - 1$, we can verify that $a > 0$ and $x < \frac{1}{a}$, therefore we have

$$\begin{aligned} T(\sqrt{2}-1) &= T\left(\frac{(\sqrt{2}-1)+(\sqrt{2}-1)}{1-(\sqrt{2}-1)\cdot(\sqrt{2}-1)}\right) - T(\sqrt{2}-1), \\ T(\sqrt{2}-1) &= T\left(\frac{2\sqrt{2}-2}{1-\left(2+1-2\sqrt{2}\right)}\right) - T(\sqrt{2}-1), \\ T(\sqrt{2}-1) &= T\left(\frac{2\sqrt{2}-2}{2\sqrt{2}-2}\right) - T(\sqrt{2}-1), \\ 2T(\sqrt{2}-1) &= T(1). \end{aligned}$$

In $T(x) = T_\infty - T(x^{-1})$, let $x = 1$. We have

$$T(1) = T_\infty - T(1),$$
$$2T(1) = T_\infty.$$

Therefore, $T(\sqrt{2} - 1) = \frac{1}{4} T_\infty$, as desired.

2017.3 Question 7

$$\begin{aligned}\frac{x^2}{a^2}+\frac{y^2}{b^2}&=\left(\frac{1-t^2}{1+t^2}\right)^2+\left(\frac{2t}{1+t^2}\right)^2\\&=\frac{\left(1-t^2\right)^2+\left(2t\right)^2}{\left(1+t^2\right)^2}\\&=\frac{1-2t^2+t^4+4t^2}{\left(1+t^2\right)^2}\\&=\frac{1+2t^2+t^4}{\left(1+t^2\right)^2}\\&=\frac{\left(1+t^2\right)^2}{\left(1+t^2\right)^2}\\&=1\end{aligned}$$

as desired, so T lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

1. The gradient of L must satisfy that

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=\frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t}\\&=\frac{b}{a}\cdot\frac{\mathrm{d}\left(2t/(1+t^2)\right)/\,\mathrm{d}t}{\mathrm{d}\left((1-t^2)/(1+t^2)\right)/\,\mathrm{d}t}\\&=\frac{b}{a}\cdot\frac{2\cdot(1+t^2)-2t\cdot 2t}{-2t\cdot(1+t^2)-(1-t^2)\cdot 2t}\\&=\frac{b}{a}\cdot\frac{2+2t^2-4t^2}{-2t-2t^3-2t+2t^3}\\&=\frac{b}{a}\cdot\frac{1-t^2}{-2t}.\end{aligned}$$

Therefore, we have a general point $(X,Y)\in L$ satisfy that

$$\begin{aligned}Y-\frac{2bt}{1+t^2}&=\frac{b}{a}\cdot\frac{1-t^2}{-2t}\cdot\left(X-\frac{a(1-t^2)}{1+t^2}\right)\\(1+t^2)Y-2bt&=\frac{b}{a}\cdot\frac{1-t^2}{-2t}\cdot\left((1+t^2)X-a(1-t^2)\right)\\(-2at)(1+t^2)Y-(-2at)(2bt)&=b\cdot(1-t^2)\cdot\left((1+t^2)X-a(1-t^2)\right)\\(-2at)(1+t^2)Y&=b(1-t^2)(1+t^2)X-ab(1-t^2)^2-4abt^2\\(-2at)(1+t^2)Y&=b(1-t^2)(1+t^2)X-ab(1+t^2)^2\\-2atY&=b(1-t^2)X-ab(1+t^2)\\ab(1+t^2)-2atY-b(1-t^2)X&=0\\(a+X)bt^2-2aYt+b(a-X)&=0\end{aligned}$$

as desired.

Now if we fix X,Y and solve for t, there are two solutions to this quadratic equation exactly when

$$\begin{aligned}(2aY)^2-4(a+X)b\cdot b(a-X)&>0\\(aY)^2-(a+X)(a-X)b^2&>0\\a^2Y^2&>(a^2-X^2)b^2,\end{aligned}$$

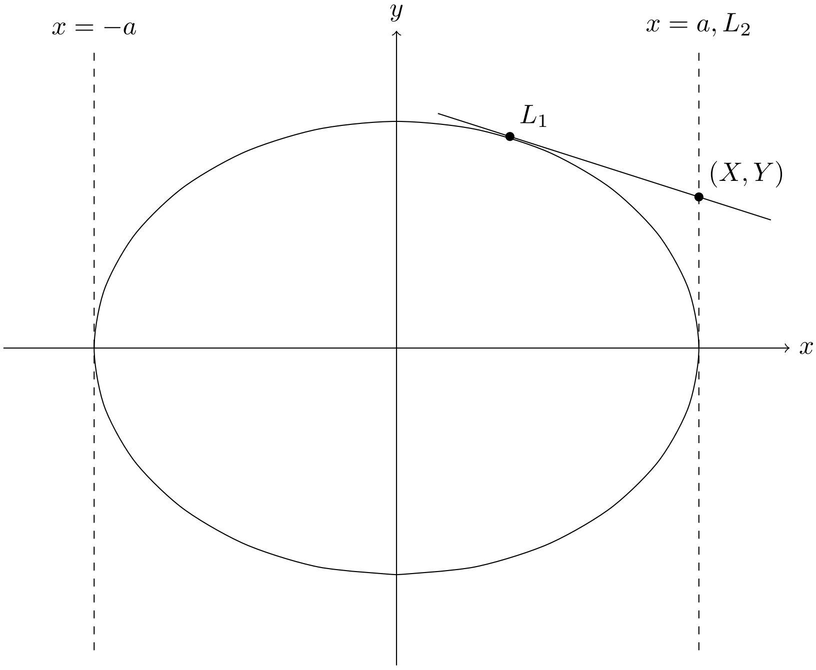
which corresponds to two distinct points on the ellipse.

Since $a^2Y^2 > (a^2 - X^2)b^2$, we have $\frac{Y^2}{b^2} > 1 - \frac{X^2}{a^2}$ by dividing through a^2b^2 on both sides, i.e.

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} > 1,$$

which means when the point (X, Y) lies outside the ellipse.

This also holds when $X^2 = a^2$, i.e. when the point (X, Y) lies on the pair of lines $X = \pm A$. Here, the condition is simply $a^2Y^2 > 0$, which gives $Y \neq 0$. One of the tangents will be the vertical line $X = \pm A$ (whichever one the point lies on), and the other one as a non-vertical (as shown when $X = a$, the tangents being L_1 and L_2).



2. By Vieta's Theorem, we have

$$pq = \frac{b(a - X)}{b(a + X)} \implies (a + X)pq = a - X,$$

as desired, and

$$p + q = -\frac{-2aY}{(a + X)b} = \frac{2aY}{(a + X)b}.$$

Let $X = 0$ for the equation in L,

$$\begin{aligned} abt^2 - 2aYt + ba &= 0 \\ bt^2 - 2Yt + b &= 0 \\ Y &= \frac{b(1 + t^2)}{2t}. \end{aligned}$$

Therefore,

$$\begin{aligned} y_1 + y_2 &= \frac{b(1 + p^2)}{2p} + \frac{b(1 + q^2)}{2q} \\ &= \frac{b\left[(1 + p^2)q + (1 + q^2)p\right]}{2pq} \\ &= 2b, \end{aligned}$$

therefore we have

$$4pq = (1 + p^2)q + (1 + q^2)p = (p + q)(1 + pq).$$

Therefore,

$$
\begin{aligned}
4 \cdot \frac{a-X}{a+X} &= \frac{2aY}{(a+X)b} \cdot \frac{2a}{a+X} \\
a - X &= \frac{a^2 Y}{b(a+X)} \\
(a-X)(a+X)b &= a^2 Y \\
(a^2 - X^2)b &= a^2 Y \\
1 - \frac{X^2}{a^2} &= \frac{Y}{b} \\
\frac{X^2}{a^2} + \frac{Y}{b} &= 1,
\end{aligned}
$$

as desired.

2017.3 Question 8

We have

$$\begin{aligned}\sum_{m=1}^{n} a_m(b_{m+1}-b_m) &= \sum_{m=1}^{n} a_m b_{m+1} - \sum_{m=1}^{n} a_m b_m \\ &= -\sum_{m=0}^{n-1} b_{m+1}a_{m+1} + \sum_{m=1}^{n} b_{m+1}a_m \\ &= -\sum_{m=1}^{n} b_{m+1}a_{m+1} + \sum_{m=1}^{n} b_{m+1}a_m + a_{n+1}b_{n+1} - a_1 b_1 \\ &= a_{n+1}b_{n+1} - a_1 b_1 - \sum_{m=1}^{n} b_{m+1}(a_{m+1}-a_m),\end{aligned}$$

as desired.

1. Let $a_m = 1$. On one hand, we have

$$\begin{aligned}\sum_{m=1}^{n} a_m(b_{m+1}-b_m) &= \sum_{m=1}^{n} [\sin(m+1)x - \sin mx] \\ &= \sum_{m=1}^{n} 2\cos\left(\frac{(m+1)x+mx}{2}\right)\sin\left(\frac{(m+1)x-mx}{2}\right) \\ &= 2\sum_{m=1}^{n} \cos\left(m+\frac{1}{2}\right)x \sin\frac{x}{2} \\ &= 2\sin\frac{x}{2}\sum_{m=1}^{n}\cos\left(m+\frac{1}{2}\right)x.\end{aligned}$$

   On the other hand, we have

$$\begin{aligned}\sum_{m=1}^{n} a_m(b_{m+1}-b_m) &= a_{n+1}b_{n+1} - a_1 b_1 - \sum_{m=1}^{n} b_{m+1}(a_{m+1}-a_m) \\ &= \sin(n+1)x - \sin x.\end{aligned}$$

   Therefore, by rearranging, we have

$$\sum_{m=1}^{n}\cos\left(m+\frac{1}{2}\right)x = \frac{1}{2}\left[\sin(n+1)x - \sin x\right]\operatorname{cosec}\frac{1}{2}x$$

   as desired.

2. Let $a_m = m$, and let $b_m = \cos\left(m-\frac{1}{2}\right)x$. We have the identity

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right).$$

   Therefore, we have

$$\begin{aligned}\sum_{m=1}^{n} a_m(b_{m+1}-b_m) &= \sum_{m=1}^{n} m\cdot\left[\cos\left(m+\frac{1}{2}\right)x - \cos\left(m-\frac{1}{2}\right)x\right] \\ &= \sum_{m=1}^{n} -2m\sin mx\sin\frac{1}{2}x \\ &= -2\sin\frac{1}{2}x\sum_{m=1}^{n} m\sin mx,\end{aligned}$$

and

$$\begin{aligned}
&\sum_{m=1}^{n} a_m(b_{m+1} - b_m) \\
&= a_{n+1}b_{n+1} - a_1b_1 - \sum_{m=1}^{n} b_{m+1}(a_{m+1} - a_m) \\
&= (n+1)\cos\left(n + \frac{1}{2}\right)x - 1 \cdot \cos\frac{1}{2}x - \sum_{m=1}^{n} \cos\left(m + \frac{1}{2}\right)x \cdot 1 \\
&= (n+1)\cos\left(n + \frac{1}{2}\right)x - \cos\frac{1}{2}x - \sum_{m=1}^{n} \cos\left(m + \frac{1}{2}\right)x \\
&= (n+1)\cos\left(n + \frac{1}{2}\right)x - \cos\frac{1}{2}x - \frac{1}{2}\left(\sin(n+1)x - \sin x\right)\operatorname{cosec}\frac{1}{2}x \\
&= \frac{1}{2}\operatorname{cosec}\frac{1}{2}x\left[2(n+1)\cos\left(n + \frac{1}{2}\right)x\sin\frac{1}{2}x - 2\cos\frac{1}{2}x\sin\frac{1}{2}x - \left(\sin(n+1)x - \sin x\right)\right] \\
&= \frac{1}{2}\operatorname{cosec}\frac{1}{2}x\left[(n+1)\left(\sin(n+1)x - \sin nx\right) - (\sin x - \sin 0) - (\sin(n+1)x - \sin x)\right] \\
&= \frac{1}{2}\operatorname{cosec}\frac{1}{2}x\left[n\sin(n+1)x - (n+1)\sin nx\right].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
-2\sin\frac{1}{2}x\sum_{m=1}^{n} m\sin mx &= \frac{1}{2}\operatorname{cosec}\frac{1}{2}x\left[n\sin(n+1)x - (n+1)\sin nx\right] \\
\sum_{m=1}^{n} m\sin mx &= -\frac{1}{4}\operatorname{cosec}^2\frac{1}{2}x\left[n\sin(n+1)x - (n+1)\sin nx\right],
\end{aligned}$$

and therefore, $p = -\frac{1}{4}n$, $q = \frac{1}{4}(n+1)$.

2017.3 Question 12

1. First, note that

$$\begin{aligned}
1 &= \sum_{x,y=1}^{x=n} \mathrm{P}(X = x, Y = y) \\
&= \sum_{x=1}^{n}\sum_{y=1}^{n} k(x+y) \\
&= \sum_{x=1}^{n}\sum_{y=1}^{n} (kx+ky) \\
&= \sum_{x=1}^{n} \left(n \cdot kx + k\sum_{y=1}^{n} y \right) \\
&= nk\sum_{x=1}^{n} x + nk\sum_{y=1}^{n} y \\
&= n^2(n+1)k
\end{aligned}$$

   Therefore, $k = \frac{1}{n^2(n+1)}$

$$\begin{aligned}
\mathrm{P}(X = x) &= \sum_{y=1}^{n} \mathrm{P}(X = x, Y = y) \\
&= \sum_{y=1}^{n} k(x+y) \\
&= nkx + k\sum_{y=1}^{n} y \\
&= nkx + \frac{kn(n+1)}{2} \\
&= \frac{x}{n(n+1)} + \frac{1}{2n} \\
&= \frac{2x+n+1}{2n(n+1)},
\end{aligned}$$

   as desired.
   By symmetry, $\mathrm{P}(Y = y) = \frac{2y+n+1}{2n(n+1)}$.
   We have

$$\mathrm{P}(X = x) \cdot \mathrm{P}(Y = y) = \frac{(2x+n+1)(2y+n+1)}{4n^2(n+1)^2}.$$

   But $\mathrm{P}(X = x, Y = y) = \frac{x+y}{n^2(n+1)}$ is not equal to this. So X and Y are not independent.

2. By definition,

$$\mathrm{Cov}(X, Y) = \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y).$$

We have

$$\begin{aligned}
\mathrm{E}(X) = \mathrm{E}(Y) &= \sum_{t=1}^{n} t \cdot \mathrm{P}(X = t) \\
&= \sum_{t=1}^{n} \frac{t \cdot (2t + n + 1)}{2n(n+1)} \\
&= \frac{1}{n(n+1)} \sum_{t=1}^{n} t^2 + \frac{1}{2n} \sum_{t=1}^{n} t \\
&= \frac{n(n+1)(2n+1)}{6n(n+1)} + \frac{n(n+1)}{4n} \\
&= \frac{2n+1}{6} + \frac{n+1}{4} \\
&= \frac{4n + 2 + 3n + 3}{12} \\
&= \frac{7n+5}{12},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}(XY) &= \sum_{x,y=1}^{n} xy \cdot \mathrm{P}(X = x, Y = y) \\
&= \sum_{x=1}^{n} \sum_{y=1}^{n} \frac{xy(x+y)}{n^2(n+1)} \\
&= \frac{1}{n^2(n+1)} \sum_{x=1}^{n} \sum_{y=1}^{n} xy(x+y) \\
&= \frac{1}{n^2(n+1)} \sum_{x=1}^{n} \sum_{y=1}^{n} (x^2 y + xy^2) \\
&= \frac{1}{n^2(n+1)} \left[\sum_{x=1}^{n} x^2 \sum_{y=1}^{n} y + \sum_{x=1}^{n} x \sum_{y=1}^{n} y^2 \right] \\
&= \frac{1}{n^2(n+1)} \cdot 2 \cdot \frac{n(n+1)(2n+1)}{6} \cdot \frac{n(n+1)}{2} \\
&= \frac{(2n+1)(n+1)}{6}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Cov}(X, Y) &= \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y) \\
&= \frac{(2n+1)(n+1)}{6} - \frac{(7n+5)^2}{144} \\
&= \frac{48n^2 + 72n + 24}{144} - \frac{49n^2 + 70n + 25}{144} \\
&= \frac{-n^2 + 2n - 1}{144} \\
&= -\frac{(n-1)^2}{144} \\
&< 0,
\end{aligned}$$

as desired.

2017.3 Question 13

We have

$$\begin{aligned} V(x) &= \mathrm{E}[(X-x)^2] \\ &= \mathrm{E}(X^2 - 2xX + x^2) \\ &= \mathrm{E}(X^2) - 2x\,\mathrm{E}(X) + x^2 \\ &= \sigma^2 + \mu^2 - 2x\mu + x^2. \end{aligned}$$

Therefore, if $Y = V(X)$, then

$$\begin{aligned} \mathrm{E}(Y) &= \mathrm{E}(V(X)) \\ &= \mathrm{E}(\sigma^2 + \mu^2 - 2X\mu + X^2) \\ &= \sigma^2 + \mu^2 - 2\mu\,\mathrm{E}(X) + \mathrm{E}(X^2) \\ &= \sigma^2 + \mu^2 - 2\mu^2 + \mu^2 + \sigma^2 \\ &= 2\sigma^2. \end{aligned}$$

Let $X \sim U[0,1]$, we have $\mu = \mathrm{E}(X) = \frac{1}{2}$, and $\sigma^2 = \mathrm{Var}(X) = \frac{1}{12}$. Therefore,

$$\begin{aligned} V(x) &= \frac{1}{12} + \frac{1}{4} - x + x^2 \\ &= x^2 - x + \frac{1}{3}. \end{aligned}$$

The c.d.f. of X is F, defined as

$$\mathrm{P}(X \leq x) = F(x) = \begin{cases} 0, & x \leq 0, \\ x, & 0 < x \leq 1, \\ 1, & 1 < x \end{cases}$$

Let the c.d.f. of Y be G, we have $G(y) = \mathrm{P}(Y \leq y)$.
Since $V([0,1]) = \left[\frac{1}{12}, \frac{1}{3}\right]$, we must have $G(y) = 0$ for $y \leq \frac{1}{12}$ and $G(y) = 1$ for $y > \frac{1}{3}$.
For $y \in \left(\frac{1}{12}, \frac{1}{3}\right]$, we have

$$\begin{aligned} G(y) = \mathrm{P}(Y \leq y) &= \mathrm{P}(V(X) \leq y) \\ &= \mathrm{P}\left(\left(x - \frac{1}{2}\right)^2 + \frac{1}{12} \leq y\right) \\ &= \mathrm{P}\left(\left|x - \frac{1}{2}\right| \leq \sqrt{y - \frac{1}{12}}\right) \\ &= \mathrm{P}\left(\frac{1}{2} - \sqrt{y - \frac{1}{12}} \leq x \leq \frac{1}{2} + \sqrt{y - \frac{1}{12}}\right) \\ &= F\left(\frac{1}{2} + \sqrt{y - \frac{1}{12}}\right) - F\left(\frac{1}{2} - \sqrt{y - \frac{1}{12}}\right) \\ &= \left(\frac{1}{2} + \sqrt{y - \frac{1}{12}}\right) - \left(\frac{1}{2} - \sqrt{y - \frac{1}{12}}\right) \\ &= 2\sqrt{y - \frac{1}{12}}. \end{aligned}$$

Therefore, the p.d.f. of y, g satisfies that for $y \in \left(\frac{1}{12}, \frac{1}{3}\right]$,

$$g(y) = G'(y) = \frac{1}{\sqrt{y - \frac{1}{12}}}$$

and 0 everywhere else.