Year 2016

2016.3 Paper 3 . . . 6
2016.3.1 Question 1 . . . 7
2016.3.2 Question 2 . . . 10
2016.3.3 Question 3 . . . 12
2016.3.4 Question 4 . . . 14
2016.3.5 Question 5 . . . 16
2016.3.6 Question 6 . . . 18
2016.3.7 Question 7 . . . 20
2016.3.8 Question 8 . . . 22
2016.3.12 Question 12 . . . 25
2016.3.13 Question 13 . . . 26

2016 Paper 3

2016.3.1 Question 1 7
2016.3.2 Question 2 10
2016.3.3 Question 3 12
2016.3.4 Question 4 14
2016.3.5 Question 5 16
2016.3.6 Question 6 18
2016.3.7 Question 7 20
2016.3.8 Question 8 22
2016.3.12 Question 12 25
2016.3.13 Question 13 26

2016.3 Question 1

Notice that

$$I_n = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(x^2+2ax+b)^n} = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{((x+a)^2+(b-a^2))^n}.$$

1. Let $x + a = \sqrt{b-a^2}\tan u$. When $x \to -\infty$, $u \to -\frac{\pi}{2}$, and when $x \to +\infty$, $u \to \frac{\pi}{2}$. We have also

$$\begin{aligned}\mathrm{d}x = \mathrm{d}(x+a) &= \mathrm{d}\sqrt{b-a^2}\tan u \\ &= \sqrt{b-a^2}\,\mathrm{d}\tan u \\ &= \sqrt{b-a^2}\sec^2 u\,\mathrm{d}u.\end{aligned}$$

   Therefore, we have

$$\begin{aligned}I_1 &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(x+a)^2+(b-a^2)} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b-a^2}\sec^2 u\,\mathrm{d}u}{\left(\sqrt{b-a^2}\tan u\right)^2+(b-a^2)} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b-a^2}\sec^2 u\,\mathrm{d}u}{(b-a^2)(\tan^2 u+1)} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sec^2 u\,\mathrm{d}u}{\sqrt{b-a^2}\sec^2 u} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sqrt{b-a^2}} \\ &= \frac{\pi}{\sqrt{b-a^2}},\end{aligned}$$

   as desired.

2. Using the same substitution, we have

$$\begin{aligned}I_n &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{[(x+a)^2+(b-a^2)]^n} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{b-a^2}\sec^2 u\,\mathrm{d}u}{[(b-a^2)\sec^2 u]^n} \\ &= \frac{1}{\sqrt{b-a^2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{[(b-a^2)\sec^2 u]^{n-1}}.\end{aligned}$$

   Therefore,

$$2n(b-a^2)I_{n+1} = (2n-1)I_n,$$

   is equivalent to

$$2n\sqrt{b-a^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{[(b-a^2)\sec^2 u]^n} = (2n-1)\frac{1}{\sqrt{b-a^2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{[(b-a^2)\sec^2 u]^{n-1}}$$

   is equivalent to

$$2n(b-a^2)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{[(b-a^2)\sec^2 u]^n} = (2n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{[(b-a^2)\sec^2 u]^{n-1}}$$

   is equivalent to

$$2n\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n} u} = (2n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n-2} u}.$$

Notice that

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n-2} u} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sec^2 u \,\mathrm{d}u}{\sec^{2n} u} \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}\tan u}{\sec^{2n} u} \\
&= \lim_{\substack{a\to\frac{\pi}{2} \\ b\to-\frac{\pi}{2}}} \left[\frac{\tan u}{\sec^{2n} u}\right]_b^a - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan u \,\mathrm{d}\sec^{-2n} u \\
&= \lim_{\substack{a\to\frac{\pi}{2} \\ b\to-\frac{\pi}{2}}} \left[\sin u \cos^{2n-1} u\right]_b^a - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -2n \sec u \tan u \sec^{-2n-1} u \tan u \,\mathrm{d}u \\
&= 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\tan^2 u \,\mathrm{d}u}{\sec^{2n} u} \\
&= 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\sec^2 u - 1) \,\mathrm{d}u}{\sec^{2n} u} \\
&= 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n-2} u} - 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n} u}.
\end{aligned}$$

This means

$$(2n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n-2} u} = 2n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}u}{\sec^{2n} u},$$

which is exactly what was desired.

3. Proof by induction:

- **Base Case.** When $n = 1$,

$$\text{LHS} = I_1 = \frac{\pi}{\sqrt{b-a^2}},$$

$$\text{RHS} = \frac{\pi}{2^{2\cdot 1-2}(b-a^2)^{1-\frac{1}{2}}} \binom{2\cdot 1-2}{1-1} = \frac{\pi}{\sqrt{b-a^2}} \binom{0}{0} = \frac{\pi}{\sqrt{b-a^2}}.$$

- **Induction Hypothesis.** Assume for some $n = k \in \mathbb{N}$, we have

$$I_n = \frac{\pi}{2^{2n-2}(b-a^2)^{n-\frac{1}{2}}} \binom{2n-2}{n-1}.$$

- **Induction Step.** When $n = k+1$,

$$\begin{aligned}
I_n &= I_{k+1} \\
&= \frac{2k+1}{2(k+1)(b-a^2)} I_k \\
&= \frac{2k+1}{2(k+1)(b-a^2)} \cdot \frac{\pi}{2^{2k-2}(b-a^2)^{k-\frac{1}{2}}} \binom{2k-2}{k-1} \\
&= \frac{\pi}{2^{2k}(b-a^2)^{k+\frac{1}{2}}} \frac{(2k-2)!}{(k-1)!(k-1)!} \frac{(2k+1)(2k+2)}{(k+1)^2} \\
&= \frac{\pi}{2^{2k}(b-a^2)^{k+\frac{1}{2}}} \frac{2k!}{k!k!} \\
&= \frac{\pi}{2^{2k}(b-a^2)^{k+\frac{1}{2}}} \binom{2k}{k} \\
&= \frac{\pi}{2^{2n-2}(b-a^2)^{n-\frac{1}{2}}} \binom{2n-2}{n-1}.
\end{aligned}$$

Therefore, by the principle of mathematical induction, for $n \in \mathbb{N}$,

$$I_n = \frac{\pi}{2^{2n-2}(b-a^2)^{n-\frac{1}{2}}}\binom{2n-2}{n-1},$$

as desired.

2016.3 Question 2

1. For $y^2 = 4ax$, we have $x = \frac{y^2}{4a}$, and therefore

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{2y}{4a}.$$

Therefore, the normal through Q, l_Q satisfies that

$$l_Q : x - aq^2 = -\frac{4a}{2 \cdot 2aq} \cdot (y - 2aq)\,,$$

i.e.

$$l_Q : q(x - aq^2) = -(y - 2aq)\,.$$

Since $P \in l_Q$, we must have

$$\begin{aligned} q(ap^2 - aq^2) &= -(2ap - 2aq) \\ aq(p+q)(p-q) &= -2a(p-q) \\ pq + q^2 &= -2 \\ q^2 + pq + 2 &= 0 \end{aligned}$$

as desired.

2. We also have

$$r^2 + pr + 2 = 0.$$

Since $q \neq r$, q, r are the solutions to the equation

$$x^2 + px + 2 = 0,$$

and therefore $q + r = -p, qr = 2$.
Note that the equation for QR satisfies that

$$m_{QR} = \frac{2ar - 2aq}{ar^2 - aq^2} = \frac{2}{r+q}.$$

Therefore, l_{QR} satisfies that

$$\begin{aligned} l_{QR} : y - 2aq &= \frac{2}{r+q}(x - aq^2) \\ y &= \frac{2}{r+q}\left(x - aq^2 + \frac{r+q}{2} \cdot 2aq\right) \\ y &= \frac{2}{r+q}\left(x - aq^2 + aq^2 + aqr\right) \\ y &= \frac{2}{r+q}\left(x + aqr\right) \\ y &= -\frac{2}{p}(x + 2a). \end{aligned}$$

This passes through a fixed point $(-2a, 0)$.

3. OP has equation $y = \frac{2ap}{ap^2}x$, which is $y = \frac{2x}{p}$. Therefore, since $T = OP \cap QR$, x_T must satisfy that

$$\begin{aligned} -\frac{2}{p}(x + 2a) &= \frac{2x}{p}, \\ -(x + 2a) &= x \\ x &= -a. \end{aligned}$$

Therefore, $y_T = -\frac{2a}{p}$, $T\left(-a, -\frac{2a}{p}\right)$ lies on the line $x = -a$ which is independent of p.

The distance from the x-axis to T is $\left|\frac{2a}{p}\right| = \frac{2a}{|p|}$.

Notice that since $qr = 2$, q and r must take the same parity, and therefore $|p| = |q| + |r|$. By the AM-GM inequality, we have

$$|q| + |r| \geq 2\sqrt{|q| \cdot |r|} = 2\sqrt{2},$$

with the equal sign holding if and only if $|q| = |r|$, $q = r$, which is impossible.

Therefore, $|p| > 2\sqrt{2}$ and therefore $\frac{2a}{|p|} < \sqrt{2}$ as desired.

2016.3 Question 3

1. We have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\frac{e^x P(x)}{Q(x)} &= \frac{Q(x)\left[e^x P'(x) + e^x P(x)\right] - Q'(x)e^x P(x)}{Q(x)^2} \\ &= e^x \frac{\left[Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)\right]}{Q(x)^2} \\ &= e^x \frac{x^3-2}{(x+1)^2}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\frac{\left[Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)\right]}{Q(x)^2} &= \frac{x^3-2}{(x+1)^2} \\ (x+1)^2\left[Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)\right] &= Q(x)^2\left(x^3-2\right).\end{aligned}$$

If we plug in $x=-1$ on both sides, we have $\mathrm{LHS}=0$ and $\mathrm{RHS}=Q(-1)^2\cdot(-3)$.
Therefore, $Q(-1)^2=0$, $Q(-1)=0$.
Since $Q(x)\in\mathbb{P}[x]$, we must have

$$(x+1) \mid Q(x)$$

as desired.
Therefore, $\deg Q \geq 1$, $\deg \mathrm{RHS} = 3 + 2\deg Q$.
If $\deg P = -\infty$, $P(x)=0$,$\mathrm{LHS}=0$ which is impossible.
If $\deg P = 0$, $P(x) = C \in \mathbb{R}\setminus\{0\}$, $\mathrm{LHS} = C(x+1)^2 Q(x)$, $\deg \mathrm{LHS} = \deg q + 2$, which is impossible.
Therefore, we have $\deg P' = \deg P - 1$. Hence,

$$\deg Q(x)P'(x) = \deg P'(x)Q(x) = \deg P + \deg Q - 1,$$

and

$$\deg Q(x)P(x) = \deg P + \deg Q.$$

Therefore,

$$\deg \mathrm{LHS} = 2 + \deg P + \deg Q = \deg \mathrm{RHS},$$

which gives

$$\deg P = \deg Q + 1,$$

as desired.
When $Q(x) = x+1$, let $P(x) = ax^2+bx+c$ where $a \neq 0$. We have $P'(x) = 2ax+b$. Therefore,

$$\begin{aligned}(x+1)^2\left[Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)\right] &= Q(x)^2\left(x^3-2\right) \\ Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x) &= x^3-2 \\ (x+1)(2ax+b) + (x+1)(ax^2+bx+c) - (ax^2+bx+c) &= x^3-2 \\ (x+1)(2ax+b) + x(ax^2+bx+c) &= x^3-2 \\ ax^3 + (2a+b)x^2 + (2a+b+c)x + b &= x^3-2.\end{aligned}$$

This solves to $(a,b,c) = (1,-2,0)$. Therefore, $P(x) = x^2-2x$.

2. In this case, we must have that

$$(x+1)\left[Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)\right] = Q(x)^2.$$

Therefore, $Q(x) = (x+1)R(x)$ for some $R(x)\in\mathbb{P}[x]$. We may assume $P(-1)\neq 0$.
Hence, $Q'(x) = (x+1)R'(x) + R(x)$

Plugging this in gives us

$$(x+1)R(x)P'(x) + (x+1)R(x)P(x) - [(x+1)R'(x) + R(x)]\,P(x) = (x+1)R(x)^2,$$

which simplifies to

$$(x+1)\,[R(x)P'(x) + R(x)P(x) - R'(x)P(x)] - R(x)P(x) = (x+1)R(x)^2.$$

Let $x = -1$, and we can see $x + 1$ divides $R(x)$, since $x + 1$ can't divide $P(x)$.
Therefore, let $R(x) = (x+1)S(x)$, therefore $R'(x) = S(x) + (x+1)S'(x)$.
This gives

$$(x+1)S(x)\,[P'(x) + P(x)] - [S(x) + (x+1)S'(x)]\,P(x) - S(x)P(x) = (x+1)^2S(x)^2,$$

which simplifies to

$$(x+1)\,[S(x)P'(x) + S(x)P(x) - S'(x)P(x)] - 2S(x)P(x) = (x+1)^2S(x)^2.$$

Therefore, we can see that $x + 1$ divides $S(x)$ by similar reasons.

Repeating this, we can conclude that there are arbitrarily many factors of $x + 1$ in $Q(x)$ (proof by infinite descent), which is impossible.

Formally speaking, let $Q(x) = (x+1)^nT(x)$ where $T(-1) \neq 0$, $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} Q'(x) &= n(x+1)^{n-1}T(x) + (x+1)^nT'(x) \\ &= (x+1)^{n-1}\,[nT(x) + (x+1)T'(x)]\,. \end{aligned}$$

Therefore,

$$(x+1)\,[Q(x)P'(x) + Q(x)P(x) - Q'(x)P(x)] = Q(x)^2$$

simplifies to

$$(x+1)^{n+1}T(x)\,[P'(x) + P(x)] - (x+1)^n\,[nT(x) + (x+1)T'(x)]\,P(x) = (x+1)^{2n}T(x)^2,$$

which further simplifies to

$$(x+1)\,[T(x)P'(x) + T(x)P(x) - T'(x)P(x)] - nT(x)P(x) = (x+1)^nT(x)^2.$$

Now, let $x = -1$, we have that $nT(-1)P(-1) = 0$. But $n \neq 0$, $T(-1) \neq 0$, $P(-1) \neq 0$, which gives a contradiction.

Therefore, such P and Q do not exist.

2016.3 Question 4

1. Notice that
$$\frac{1}{1+x^r}-\frac{1}{1+x^{r+1}}=\frac{x^{r+1}-x^r}{(1+x^r)(1+x^{r+1})}=\frac{x^r(x-1)}{(1+x^r)(1+x^{r+1})}.$$
Therefore, we have
$$\begin{aligned}\sum_{r=1}^{N}\frac{x^r}{(1+x^r)(1+x^{r+1})}&=\sum_{r=1}^{N}\frac{1}{x-1}\left[\frac{1}{1+x^r}-\frac{1}{1+x^{r+1}}\right]\\&=\frac{1}{x-1}\sum_{r=1}^{N}\left[\frac{1}{1+x^r}-\frac{1}{1+x^{r+1}}\right]\\&=\frac{1}{x-1}\left[\frac{1}{1+x}-\frac{1}{1+x^{n+1}}\right].\end{aligned}$$
For $|x|<1$, as $n\to\infty$, $x^{n+1}\to 0$. Therefore,
$$\begin{aligned}\sum_{r=1}^{\infty}\frac{x^r}{(1+x^r)(1+x^{r+1})}&=\frac{1}{x-1}\left[\frac{1}{1+x}-1\right]\\&=\frac{1}{x-1}\cdot\frac{-x}{1+x}\\&=\frac{x}{1-x^2}\end{aligned}$$
as desired.

2. Notice that
$$\begin{aligned}\operatorname{sech}(ry)\operatorname{sech}((r+1)y)&=\frac{2}{e^{ry}+e^{-ry}}\cdot\frac{2}{e^{(r+1)y}+e^{-(r+1)y}}\\&=\frac{4e^{-ry-(r+1)y}}{(1+e^{-2ry})\left(1+e^{-2(r+1)y}\right)}\\&=4e^{-y}\frac{e^{-2ry}}{(1+e^{-2ry})\left(1+e^{-2(r+1)y}\right)}.\end{aligned}$$
Let $x=e^{-2y}$. We have
$$\operatorname{sech}(ry)\operatorname{sech}((r+1)y)=4e^{-y}\frac{x^r}{(1+x^r)(1+x^{r+1})}.$$
When $y>0$, $x=e^{-2y}\in(0,1)$. Therefore,
$$\begin{aligned}\sum_{r=1}^{\infty}\operatorname{sech}(ry)\operatorname{sech}((r+1)y)&=4e^{-y}\frac{e^{-2y}}{1-e^{-4y}}\\&=2e^{-y}\frac{2}{e^{2y}-e^{-2y}}\\&=2e^{-y}\operatorname{cosech}(2y)\end{aligned}$$
as desired.

Notice that for all $x\in\mathbb{R}$, $\cosh x=\cosh(-x)$, therefore $\operatorname{sech} x=\operatorname{sech}(-x)$.

Therefore,

$$\begin{aligned}
&\sum_{r=-\infty}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y)\\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) + \sum_{r=-\infty}^{0} \operatorname{sech}(ry)\operatorname{sech}((r+1)y)\\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) + \sum_{r=0}^{+\infty} \operatorname{sech}(-ry)\operatorname{sech}((-r+1)y)\\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) + \sum_{r=0}^{+\infty} \operatorname{sech}(ry)\operatorname{sech}((r-1)y)\\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) + \sum_{r=2}^{+\infty} \operatorname{sech}(ry)\operatorname{sech}((r-1)y) + \operatorname{sech}(y)\operatorname{sech}(0) + \operatorname{sech}(0)\operatorname{sech}(-y)\\
&= \sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) + \sum_{r=1}^{+\infty} \operatorname{sech}((r+1)y)\operatorname{sech}(ry) + 2\operatorname{sech} y\\
&= 4e^{-y}\operatorname{cosech}(2y) + 2\operatorname{sech} y\\
&= \frac{4e^{-y}}{\sinh 2y} + \frac{2}{\cosh y}\\
&= \frac{2e^{-y}}{\sinh y\cosh y} + \frac{2}{\cosh y}\\
&= \frac{2e^{-y} + 2\sinh y}{\sinh y\cosh y}\\
&= \frac{2e^{-y} + e^{y} - e^{-y}}{\sinh y\cosh y}\\
&= \frac{e^{y} - e^{-y}}{\sinh y\cosh y}\\
&= \frac{2\cosh y}{\sinh y\cosh y}\\
&= 2\operatorname{cosech} y.
\end{aligned}$$

2016.3 Question 5

1. By the binomial theorem, we have

$$(1+x)^{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} x^k.$$

If we let $x = 1$, we have

$$2^{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k}.$$

Since $\binom{2m+1}{m}$ is a part of the sum, and all the other terms are positive, and there are other terms which are not $\binom{2m+1}{m}$ (e.g. $\binom{2m+1}{0} = 1$), we therefore must have

$$\binom{2m+1}{m} < 2^{2m+1}.$$

2. Notice that

$$\begin{aligned}\binom{2m+1}{m} &= \frac{(2m+1)!}{m!(m+1)!} \\ &= \frac{(2m+1)(2m)(2m-1)\cdots(m+2)}{m!}\end{aligned}$$

A number theory argument follows. First, notice that all terms in the product $P_{m+1,2m+1}$ are within the numerator. Therefore, we must have

$$P_{m+1,2m+1} \mid (2m+1)(2m)(2m-1)\cdots(m+2).$$

Next, since all the terms in the product are primes, none of the terms will therefore have factors between 1 and m. This means that

$$\gcd\left(P_{m+1,2m+1}, m!\right) = 1,$$

i.e. $P_{m+1,2m+1}$ are coprime.

Therefore, given that $\binom{2m+1}{m} = \frac{(2m+1)(2m)(2m-1)\cdots(m+2)}{m!}$ is an integer, we must therefore have

$$P_{m+1,2m+1} \mid \binom{2m+1}{m},$$

and hence

$$P_{m+1,2m+1} \leq \binom{2m+1}{m} < 2^{2m},$$

as desired.

3. Notice that

$$\begin{aligned}P_{1,2m+1} &= P_{1,m+1} \cdot P_{m+1,2m+1} \\ &< 4^{m+1} \cdot 2^{2m} \\ &= 4^{m+1} \cdot 4^m \\ &= 4^{2m+1},\end{aligned}$$

as desired.

4. First we look at the base case when $n = 2$.

   $P_{1,2} = 2$, $4^2 = 16$, the original statement holds when $n = 2$.

   Now, we use strong induction. Suppose the statement holds up to some $n = k \geq 2$.

   If $k = 2m$ is even, the induction step for $2m \to 2m + 1$ is already shown in the previous part.

   If $k = 2m + 1$ is odd, we must have that $k + 1$ is even. The only even prime is 2, but since $k \geq 2$, $k + 1 \neq 2$, and $k + 1$ must be composite.

   Therefore, $P_{1,k+1} = P_{1,k} < 4^k < 4^{k+1}$. This completes the induction step.

   Therefore, by strong induction, the statement $P_{1,n} < 4^n$ holds for all $n \geq 2$.

2016.3 Question 6

- In the case where $B > A > 0$ or $-B < -A < 0$, notice that

$$R \cosh(x + \gamma) = R \cosh x \cosh \gamma + R \sinh x \sinh \gamma.$$

Therefore, we would like $R \sinh \gamma = A$ and $R \cosh \gamma = B$.
Since $\cosh \gamma^2 - \sinh \gamma^2 = 1$, we have $R^2 = B^2 - A^2$.
We also have $\tanh \gamma = \frac{A}{B}$, and therefore $\gamma = \operatorname{artanh} \frac{A}{B}$.
Notice that $\cosh \gamma > 0$, so R must have the same sign as B.

  - If $B > A > 0$, $R = \sqrt{B^2 - A^2}$.
  - If $B < -A < 0$, $R = -\sqrt{B^2 - A^2}$.

- In the case where $-A < B < A$, notice that

$$R \sinh(x + \gamma) = R \sinh \gamma \cosh x + R \cosh \gamma \sinh x.$$

Therefore, we would like $R \cosh \gamma = A$ and $R \sinh \gamma = B$.
Since $\cosh \gamma^2 - \sinh \gamma^2 = 1$, we have $R^2 = B^2 - A^2$.
We also have $\tanh \gamma = \frac{B}{A}$, and therefore $\gamma = \operatorname{artanh} \frac{B}{A}$.
Notice that $\cosh \gamma > 0$, so R will have the same sign as A, and hence $R = \sqrt{A^2 - B^2}$.

- When $B = A$, we have

$$\begin{aligned} A \sinh x + B \cosh x &= A \frac{e^x - e^{-x}}{2} + A \frac{e^x + e^{-x}}{2} \\ &= Ae^x. \end{aligned}$$

- When $B = -A$, we have

$$\begin{aligned} A \sinh x + B \cosh x &= A \frac{e^x - e^{-x}}{2} - A \frac{e^x + e^{-x}}{2} \\ &= Ae^{-x}. \end{aligned}$$

Therefore, in conclusion,

$$A \sinh x + B \cosh x = \begin{cases} \sqrt{B^2 - A^2} \cosh\left(x + \operatorname{artanh} \frac{A}{B}\right), & 0 < A < B, \\ Ae^x, & 0 < B = A, \\ \sqrt{A^2 - B^2} \sinh\left(x + \operatorname{artanh} \frac{B}{A}\right), & -A < B < A, \\ -Ae^{-x}, & B = -A < 0, \\ -\sqrt{B^2 - A^2} \cosh\left(x + \operatorname{artanh} \frac{A}{B}\right), & -B < -A < 0. \end{cases}$$

1. We have $\operatorname{sech} x = a \tanh x + b$, and hence $1 = a \sinh x + b \cosh x$. If $b > a > 0$, we have

$$\sqrt{b^2 - a^2} \cosh\left(x + \operatorname{artanh} \frac{a}{b}\right) = 1.$$

Therefore,

$$\begin{aligned} \cosh\left(x + \operatorname{artanh} \frac{a}{b}\right) &= \frac{1}{\sqrt{b^2 - a^2}} \\ x + \operatorname{artanh} \frac{a}{b} &= \pm \operatorname{arcosh} \frac{1}{\sqrt{b^2 - a^2}} \\ x &= \pm \operatorname{arcosh} \frac{1}{\sqrt{b^2 - a^2}} - \operatorname{artanh} \frac{a}{b}, \end{aligned}$$

as desired.

2. When $a > b > 0$,
$$\sqrt{a^2 - b^2} \sinh \left(x + \operatorname{artanh} \frac{b}{a} \right) = 1.$$
Therefore,
$$\begin{aligned} \sinh \left(x + \operatorname{artanh} \frac{b}{a} \right) &= \frac{1}{\sqrt{a^2 - b^2}} \\ x + \operatorname{artanh} \frac{b}{a} &= \operatorname{arsinh} \frac{1}{\sqrt{a^2 - b^2}} \\ x &= \operatorname{arsinh} \frac{1}{\sqrt{a^2 - b^2}} - \operatorname{artanh} \frac{b}{a}. \end{aligned}$$

3. We would like to have two solutions to the equation $1 = a \sinh x + b \cosh x$.
   - $0 < a < b$, this gives
   $$x = \pm \operatorname{arcosh} \frac{1}{\sqrt{b^2 - a^2}} - \operatorname{artanh} \frac{a}{b},$$
   For this to make sense, we must have $\frac{1}{\sqrt{b^2 - a^2}} \geq 1$, and therefore $0 < \sqrt{b^2 - a^2} \leq 1$, which is $0 < b^2 - a^2 \leq 1$.
   For this to have two distinct points, we would like to have $\operatorname{arcosh} \frac{1}{\sqrt{b^2 - a^2}} \neq 0$ as well. This means $b^2 - a^2 \neq 1$.
   Therefore, in this case, this means that $a < b < \sqrt{a^2 + 1}$.
   - $b = a$, this gives $ae^x = 1$, which gives a unique solution $x = -\ln a$.
   - $-a < b < a$, this gives
   $$\sqrt{A^2 - B^2} \sinh \left(x + \operatorname{artanh} \frac{B}{A} \right) = 1,$$
   which can only give the solution $x = \operatorname{arsinh} \frac{1}{\sqrt{A^2 - B^2}} - \operatorname{artanh} \frac{B}{A}$.
   - $b = -a$, this gives $-ae^{-x} = 1$, which does not have a solution.
   - $-b < -a < 0$, this gives
   $$-\sqrt{b^2 - a^2} \cosh \left(x + \operatorname{artanh} \frac{a}{b} \right) = 1,$$
   but this is impossible, since both square root and cosh are always positive.

   Therefore, the only possibility is when $a < b < \sqrt{a^2 + 1}$.

4. When they touch at a point, this will mean at this value, the number of solutions will change on both sides. This is only possible when $b = \sqrt{a^2 + 1}$.
   Therefore,
   $$x = -\operatorname{artanh} \frac{a}{\sqrt{a^2 + 1}}.$$
   Hence,
   $$\begin{aligned} y &= a \tanh x + b \\ &= -a \cdot \frac{a}{\sqrt{a^2 + 1}} + \sqrt{a^2 + 1} \\ &= \frac{-a^2 + a^2 + 1}{\sqrt{a^2 + 1}} \\ &= \frac{1}{\sqrt{a^2 + 1}}. \end{aligned}$$

2016.3 Question 7

For $\omega = \exp \frac{2\pi i}{n}$, we have for $k = 0, 1, 2, \ldots, n-1$, that $\omega^k = \exp \frac{2\pi i k}{n}$. Therefore,

$$(\omega^k)^n = \exp \frac{2\pi i k n}{n} = \exp(2\pi i k) = 1.$$

Also, notice that $\arg \omega^k = \frac{2k\pi}{n}$, which means that all ω^ks are different.

This means that $\omega^0 = 1, \omega^1 = 1, \omega^2, \ldots, \omega^{n-1}$ are exactly the n roots to the polynomial $z^n - 1$, which has leading coefficient 1.

Therefore, we must have

$$(z-1)(z-\omega)\cdots(z-\omega^{n-1}) = z^n - 1,$$

as desired.

For the following parts, W.L.O.G. let the orientation of the polygon be such that $X_k = \omega^k$.

1. Let z represent the complex number for P, we have

$$\begin{aligned}\prod_{k=0}^{n-1} |PX_k| &= \prod_{k=0}^{n-1} \left|z - \omega^k\right| \\ &= \left|\prod_{k=0}^{n-1} (z - \omega^k)\right| \\ &= |z^n - 1|.\end{aligned}$$

Since P is equidistant from X_0 and X_1, we must have that $P = r \exp\left(\frac{\pi i}{n}\right)$ for some $r \in \mathbb{R}$, where $|r| = |OP|$. Therefore, we have

$$\begin{aligned}\prod_{k=0}^{n-1} |PX_k| &= |z^n - 1| \\ &= \left|r^n \exp\left(\frac{\pi i}{2}\right) - 1\right| \\ &= |-r^n - 1| \\ &= |r^n + 1|.\end{aligned}$$

If n is even, then $r^n = |r|^n > 0$, and therefore $|r^n + 1| = r^n + 1 = |r|^n + 1 = |OP|^n + 1$ as desired.

If n is odd, and $r > 0$, then $r^n = |r|^n > 0$, and

$$\begin{aligned}\text{LHS} &= |r^n + 1| \\ &= r^n + 1 \\ &= |r|^n + 1 \\ &= |OP|^n + 1.\end{aligned}$$

When $-1 \leq r < 0$, we have $-1 \leq r^n = -|r|^n < 0$, and

$$\begin{aligned}\text{LHS} &= |r^n + 1| \\ &= r^n + 1 \\ &= -|r|^n + 1 \\ &= -|OP|^n + 1.\end{aligned}$$

When $r < -1$, we have $r^n = -|r|^n < -1$, and

$$\begin{aligned}\text{LHS} &= |r^n + 1| \\ &= -r^n - 1 \\ &= |r|^n - 1 \\ &= |OP|^n - 1.\end{aligned}$$

In summary, when n is odd, we have

$$\prod_{k=0}^{n-1} |PX_k| = \begin{cases} |OP|^n + 1, & P \text{ is in the first quadrant,} \\ -|OP|^n + 1, & P \text{ is in the third quadrant and } |OP| \leq 1, \\ |OP|^n - 1, & P \text{ is in the third quadrant and } |OP| > 1. \end{cases}$$

2. Notice that for a general point P whose complex number is z, we have

$$\begin{aligned} \prod_{k=1}^{n-1} |PX_k| &= (z-\omega)(z-\omega^2)\cdots\left(z-\omega^{n-1}\right) \\ &= \frac{z^n-1}{z-1} \\ &= 1 + z + z^2 + \cdots + z^{n-1}. \end{aligned}$$

If we let $P = X_0$, $z = 1$, and RHS $= n$, just as we desired.

2016.3 Question 8

1. If we replace x with $-x$ in the original equation, we get

$$f(-x) + (1 - (-x))f(-(-x)) = (-x)^2,$$

which simplifies to

$$f(-x) + (1 + x)f(x) = x^2$$

as desired.

Therefore, we have a pair of equations in terms of $f(x)$ and $f(-x)$:

$$\begin{cases} f(x) + (1 - x)f(-x) & = x^2 \\ (1 + x)f(x) + f(-x) & = x^2. \end{cases}$$

Multiplying the second equation by $(1 - x)$ gives us

$$(1 - x^2)f(x) + (1 - x)f(-x) = x^2(1 - x),$$

and subtracting the first equation from this

$$-x^2 f(x) = -x^3,$$

which gives $f(x) = x$.

Plugging this back, we have

$$\begin{aligned} \text{LHS} &= f(x) + (1 - x)f(-x) \\ &= x + (1 - x)(-x) \\ &= x - x + x^2 \\ &= x^2 \\ &= \text{RHS} \end{aligned}$$

which holds. Therefore, $f(x) = x$ is the solution to the functional equation.

2. For $x \neq 1$, we have

$$\begin{aligned} K(K(x)) &= \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} \\ &= \frac{(x + 1) + (x - 1)}{(x + 1) - (x - 1)} \\ &= \frac{2x}{2} \\ &= x, \end{aligned}$$

for $x \neq 1$, as desired.

The equation on g is

$$g(x) + xg(K(x)) = x,$$

and if we substitute x as $K(x)$, we have

$$g(K(x)) + K(x)g(K(K(x))) = K(x),$$

which simplifies to

$$g(K(x)) + K(x)g(x) = K(x).$$

Multiplying the second equation by x, we have

$$xK(x)g(X) + xg(K(x)) = xK(x),$$

and subtracting the first equation from this gives

$$(xK(x)-1)g(x) = x(K(x)-1),$$

which gives

$$\begin{aligned} g(x) &= \frac{x\left(K(x)-1\right)}{xK(x)-1} \\ &= \frac{x\left(\frac{x+1}{x-1}-1\right)}{x\cdot\frac{x+1}{x-1}-1} \\ &= \frac{x\left[(x+1)-(x-1)\right]}{x(x+1)-(x-1)} \\ &= \frac{2x}{x^2+1}, \end{aligned}$$

for $x \neq 1$.

If we plug this back to the original equation, we have

$$\begin{aligned} \text{LHS} &= \frac{2x}{x^2+1} + x\frac{2\cdot\frac{x+1}{x-1}}{\left(\frac{x+1}{x-1}\right)^2+1} \\ &= \frac{2x}{x^2+1} + \frac{2x\cdot(x+1)\cdot(x-1)}{(x+1)^2+(x-1)^2} \\ &= \frac{2x}{x^2+1} + \frac{2x(x^2-1)}{2x^2+2} \\ &= \frac{2x}{x^2+1} + \frac{x(x^2-1)}{x^2+1} \\ &= \frac{x^3-x+2x}{x^2+1} \\ &= \frac{x(x^2+1)}{x^2+1} \\ &= x \\ &= \text{RHS}, \end{aligned}$$

so

$$g(x) = \frac{2x}{x^2+1}$$

is the solution to the original functional equation.

3. Let $H(x) = \frac{1}{1-x}$. Notice that

$$\begin{aligned} H(H(x)) &= \frac{1}{1-\frac{1}{1-x}} \\ &= \frac{1-x}{1-x-1} \\ &= \frac{x-1}{x} \\ &= 1-\frac{1}{x} \end{aligned}$$

and

$$\begin{aligned} H(H(H(x))) &= \frac{1}{1-\left(1-\frac{1}{x}\right)} \\ &= \frac{x}{1} \\ &= x. \end{aligned}$$

Now, if we replace all the x with $\frac{1}{1-x}$, we will get

$$h\left(\frac{1}{1-x}\right)+h\left(1-\frac{1}{x}\right)=1-\frac{1}{1-x}-\left(1-\frac{1}{x}\right),$$

and doing the same replacement again gives us

$$h\left(1-\frac{1}{x}\right)+h(x)=1-\left(1-\frac{1}{x}\right)-x.$$

Summing these two equations, together with the original equation, gives us that

$$2\cdot\left[h\left(\frac{1}{1-x}\right)+h\left(1-\frac{1}{x}\right)+h(x)\right]=3-2\cdot\left[x+\frac{1}{1-x}+\left(1-\frac{1}{x}\right)\right],$$

and therefore

$$h\left(\frac{1}{1-x}\right)+h\left(1-\frac{1}{x}\right)+h(x)=\frac{3}{2}-\left[x+\frac{1}{1-x}+\left(1-\frac{1}{x}\right)\right].$$

Subtracting the second equation from this, gives that

$$\begin{aligned}h(x)&=\left(\frac{3}{2}-\left[x+\frac{1}{1-x}+\left(1-\frac{1}{x}\right)\right]\right)-\left[1-\frac{1}{1-x}-\left(1-\frac{1}{x}\right)\right]\\&=\frac{1}{2}-x.\end{aligned}$$

Plugging this back to the original equation, we have

$$\begin{aligned}\text{LHS}&=\frac{1}{2}-x+\frac{1}{2}-\frac{1}{1-x}\\&=1-x-\frac{1}{1-x}\\&=\text{RHS},\end{aligned}$$

which satisfies the original functional equation. Therefore, the original equation solves to

$$h(x)=\frac{1}{2}-x.$$

2016.3 Question 12

1. Let $X \sim \mathrm{B}(100n, 0.2)$. We have $\mu = 100n \cdot 0.2 = 20n$, and $\sigma^2 = 100n \cdot 0.2 \cdot 0.8 = 16n$.

   We have that

$$\begin{aligned}\alpha &= \mathrm{P}(16n \leq X \leq 24n) \\ &= \mathrm{P}(|(X - 20n)| \leq 4n) \\ &= \mathrm{P}(|(X - \mu)| \leq \sigma\sqrt{n}) \\ &= 1 - \mathrm{P}(|(X - \mu)| > \sigma\sqrt{n}) \\ &\geq 1 - \frac{1}{\sqrt{n}^2} \\ &= 1 - \frac{1}{n},\end{aligned}$$

   as desired, where we applied the Chebyshev Inequality for $k = \sqrt{n} > 0$.

2. Let $X \sim \mathrm{Po}(n)$. Therefore, $\mu = \mathrm{E}(X) = n$, $\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{n}$. To show the desired inequality is equivalent to showing that

$$\frac{1 + n + \frac{n^2}{2!} + \cdot + \frac{n^{2n}}{(2n!)}}{e^n} \geq 1 - \frac{1}{n}.$$

   Notice that the left-hand side is simply $\mathrm{P}(0 \leq X \leq 2n)$. By the Chebyshev Inequality, we have

$$\begin{aligned}\mathrm{LHS} &= \mathrm{P}(0 \leq X \leq 2n) \\ &= \mathrm{P}(|X - \mu| \leq n) \\ &= \mathrm{P}(|X - \mu| \leq \sqrt{n}\sigma) \\ &= 1 - \mathrm{P}(|X - \mu| > \sqrt{n}\sigma) \\ &\geq 1 - \frac{1}{n} \\ &= \mathrm{RHS},\end{aligned}$$

   as desired, where we applied the Chebyshev Inequality for $k = \sqrt{n} > 0$.

2016.3 Question 13

For a random variable X with $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$, we have

$$\kappa(X) = \frac{\mathrm{E}\left[(X-\mu)^4\right]}{\sigma^4} - 3$$

We have $Y = X - a$. Therefore, $\mathrm{E}(Y) = \mu - a$ and $\mathrm{Var}(Y) = \sigma^2$.

$$\begin{aligned}
\kappa(Y) &= \frac{\mathrm{E}\left[(Y-(\mu-a))^4\right]}{\sigma^4} - 3 \\
&= \frac{\mathrm{E}\left[((X-a)-(\mu-a))^4\right]}{\sigma^4} - 3 \\
&= \frac{\mathrm{E}\left[(X-\mu)^4\right]}{\sigma^4} - 3 \\
&= \kappa(X),
\end{aligned}$$

as desired.

1. Let $X \sim \mathrm{N}(0, \sigma^2)$, $\mu = 0$. Notice that

$$\kappa(X) = \frac{\mathrm{E}(X^4)}{\sigma^4} - 3.$$

X has p.d.f.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Therefore,

$$\mathrm{E}(X^4) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^4 \exp\left(-\frac{x^2}{2\sigma^2}\right) \mathrm{d}x.$$

Now, consider using integration by parts. Notice that

$$\mathrm{d}\exp\left(-\frac{x^2}{2\sigma^2}\right) = -\frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right) \mathrm{d}x,$$

and therefore, using integration by parts, we have

$$\begin{aligned}
&\int x^4 \exp\left(-\frac{x^2}{2\sigma^2}\right) \mathrm{d}x \\
&= -\sigma^2 \int x^3 \,\mathrm{d}\exp\left(-\frac{x^2}{2\sigma^2}\right) \\
&= -\sigma^2 \left[x^3 \exp\left(-\frac{x^2}{2\sigma^2}\right) - \int \exp\left(-\frac{x^2}{2\sigma^2}\right) \mathrm{d}(x^3)\right] \\
&= 3\sigma^2 \int x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right) \mathrm{d}x - \sigma^2 x^3 \exp\left(-\frac{x^2}{2\sigma^2}\right).
\end{aligned}$$

Therefore, considering the definite integral, we have

$$\begin{aligned}
\mathrm{E}(X^4) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^4 \exp\left(-\frac{x^2}{2\sigma^2}\right) \mathrm{d}x \\
&= \frac{\sigma}{\sqrt{2\pi}} \left[3\int_{-\infty}^{+\infty} x^2 \exp\left(-\frac{x^2}{2\sigma^2}\right) \mathrm{d}x - \left[x^3 \exp\left(-\frac{x^2}{2\sigma^2}\right)\right]_{-\infty}^{+\infty}\right] \\
&= \frac{\sigma}{\sqrt{2\pi}} \left[3 \cdot \sigma\sqrt{2\pi} \cdot \sigma^2 - 0\right] \\
&= 3\sigma^4.
\end{aligned}$$

Therefore,

$$\kappa(X) = \frac{\mathrm{E}(X^4)}{\sigma^4} - 3 = \frac{3\sigma^4}{\sigma^4} - 3 = 0,$$

as desired.

An alternative solution exists using generating functions.

Recall that a general normal distribution $\mathrm{N}(\mu, \sigma^2)$ has MGF

$$M(t) = \exp(\mu t + \frac{\sigma^2}{2} t^2),$$

and hence

$$\begin{aligned} M_X(t) &= \exp\left(\frac{\sigma^2}{2} t^2\right) \\ &= 1 + \left(\frac{\sigma^2}{2} t^2\right) + \frac{\left(\frac{\sigma^2}{2} t^2\right)}{2!} + \dots . \end{aligned}$$

Therefore,

$$\mathrm{E}(X^4) = M_X^{(4)}(0) = \left(\frac{\sigma^2}{2}\right)^4 \cdot 4! = 3\sigma^4,$$

and the result follows.

2. Notice that

$$\begin{aligned} &T^4 \\ &= \sum_a \binom{4}{4} Y_a^4 + \sum_{a<b} \left[\binom{4}{1,3} Y_a Y_b^3 + \binom{4}{2,2} Y_a^2 Y_b^2\right] \\ &\quad + \sum_{a<b<c} \binom{4}{1,1,2} Y_a Y_b Y_c^2 + \sum_{a<b<c<d} \binom{4}{1,1,1,1} Y_a Y_b Y_c Y_d \\ &= \sum_a Y_a^4 + \sum_{a<b} (4 Y_a Y_b^3 + 6 Y_a^2 Y_b^2) + \sum_{a<b<c} 12 Y_a Y_b Y_c^2 + \sum_{a<b<c<d} 24 Y_a Y_b Y_c Y_d, \end{aligned}$$

where

$$\binom{n}{a_1, a_2, \dots, a_k} = \frac{n!}{a_1! a_2! \dots a_k!}, \sum_{i=1}^k a_i = n$$

stands for the multinomial coefficient.

Note that $\mathrm{E}(Y_r) = 0$ for any $r = 1, 2, \dots, n$. Therefore,

$$\begin{aligned} \mathrm{E}(Y_a Y_b^3) &= \mathrm{E}(Y_a)\,\mathrm{E}(Y_b^3) = 0, \\ \mathrm{E}(Y_a Y_b Y_c^2) &= \mathrm{E}(Y_a)\,\mathrm{E}(Y_b)\,\mathrm{E}(Y_c^2) = 0, \\ \mathrm{E}(Y_a Y_b Y_c Y_d) &= \mathrm{E}(Y_a)\,\mathrm{E}(Y_b)\,\mathrm{E}(Y_c)\,\mathrm{E}(Y_d) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{E}(T^4) &= \sum_a \mathrm{E}(Y_a^4) + \sum_{a<b} 6\,\mathrm{E}(Y_a^2 Y_b^2) \\ &= \sum_{r=1}^n \mathrm{E}(Y_r^4) + 6 \sum_{r=1}^{n-1} \sum_{s=r+1}^n \mathrm{E}(Y_a^2)\,\mathrm{E}(Y_b^2), \end{aligned}$$

as desired.

3. Let $Y_i = X_i - \mu$ for $i = 1, 2, \dots, n$, and $\mu = \mathrm{E}(X), \sigma^2 = \mathrm{Var}(X) = \mathrm{Var}(Y)$ with $\mathrm{E}(Y) = 0$
Therefore, let $T = \sum_i^n Y_i = \sum_i^n X_i - n\mu$, we must have $\mathrm{E}(T) = 0$ and $\mathrm{Var}(T) = n\sigma^2$.

But since the kurtosis remains constant with shifts, we must have that $\kappa(Y_i) = \kappa$, and

$$\kappa(T) = \kappa\left[\sum_i^n X_i\right].$$

Hence, we have

$$\begin{aligned}
\kappa\left[\sum_i^n X_i\right] &= \kappa(T) \\
&= \frac{\mathrm{E}(T^4)}{(n\sigma^2)^2} - 3 \\
&= \frac{\sum_{r=1}^n \mathrm{E}(Y_r^4) + 6\sum_{r=1}^{n-1}\sum_{s=r+1}^n \mathrm{E}(Y_a^2)\,\mathrm{E}(Y_b^2)}{n^2\sigma^4} - 3 \\
&= \frac{1}{n^2}\sum_{r=1}^n \frac{\mathrm{E}(Y_r^4)}{\sigma^4} + \frac{6}{n^2}\sum_{r=1}^{n-1}\sum_{s=r+1}^n \frac{\sigma^4}{\sigma^4} - 3 \\
&= \frac{1}{n^2} n \cdot (\kappa + 3) + \frac{6}{n^2}\binom{n}{2} - 3 \\
&= \frac{\kappa}{n} + \frac{3n + 3n(n-1) - 3n^2}{n^2} \\
&= \frac{\kappa}{n} + 0 \\
&= \frac{\kappa}{n},
\end{aligned}$$

as desired.