Year 2015

2015 Paper 3

2015.3 Question 1

1. We have

$$\begin{aligned}I_n - I_{n+1} &= \int_0^\infty \frac{1}{(1+u^2)^n}\,\mathrm{d}x - \int_0^\infty \frac{1}{(1+u^2)^{n+1}}\,\mathrm{d}x\\ &= \int_0^\infty \frac{(1+u^2)-1}{(1+u^2)^{n+1}}\,\mathrm{d}x\\ &= \int_0^\infty \frac{u^2}{(1+u^2)^{n+1}}\,\mathrm{d}x.\end{aligned}$$

Notice that

$$\frac{\mathrm{d}(1+u^2)^{-n}}{\mathrm{d}x} = -\frac{2un}{(1+u^2)^{n+1}},$$

and therefore,

$$\frac{u\,\mathrm{d}x}{(1+u^2)^{n+1}} = -\frac{\mathrm{d}(1+u^2)^{-n}}{2n}.$$

Using integration by parts, we have

$$\begin{aligned}I_n - I_{n+1} &= \int_0^\infty \frac{u^2}{(1+u^2)^{n+1}}\,\mathrm{d}x\\ &= \int_0^\infty \left[-\frac{u\,\mathrm{d}(1+u^2)^{-n}}{2n}\right]\\ &= \frac{1}{2n}\left[\int_0^\infty \frac{\mathrm{d}u}{(1+u^2)^n} - \left[u\cdot(1+u^2)^{-n}\right]_0^\infty\right]\\ &= \frac{1}{2n}\left[I_n - (0-0)\right]\\ &= \frac{1}{2n}I_n,\end{aligned}$$

as desired.

Hence, we have

$$I_{n+1} = \left(1-\frac{1}{2n}\right)I_n = \frac{2n-1}{2n}I_n.$$

Notice that

$$I_1 = \int_0^\infty \frac{\mathrm{d}u}{1+u^2} = [\arctan u]_0^\infty = \frac{\pi}{2},$$

and therefore, we have

$$\begin{aligned}I_{n+1} &= \frac{2n-1}{2n}I_n\\ &= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot I_{n-1}\\ &= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\cdot I_{n-2}\\ &\vdots\\ &= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\cdots\frac{2\cdot 1-1}{2\cdot 1}\cdot I_1\\ &= \frac{(2n-1)(2n-3)(2n-5)\cdots 3\cdot 1}{(2n)\cdot(2n-2)\cdot(2n-4)\cdots 4\cdot 2}\cdot\frac{\pi}{2}.\end{aligned}$$

Let

$$\begin{aligned}A &= (2n-1)(2n-3)(2n-5)\cdots 3\cdot 1,\\ B &= (2n)\cdot(2n-2)\cdot(2n-4)\cdots 4\cdot 2.\end{aligned}$$

Notice that

$$AB = (2n)!, B = 2^n \cdot n!,$$

and therefore

$$A = \frac{(2n)!}{2^n \cdot n!},$$

and hence

$$\begin{aligned} I_{n+1} &= \frac{(2n-1)(2n-3)(2n-5)\cdots 3\cdot 1}{(2n)\cdot(2n-2)\cdot(2n-4)\cdots 4\cdot 2} \cdot \frac{\pi}{2} \\ &= \frac{(2n)!/(2^n \cdot n!)}{2^n \cdot n!} \cdot \frac{\pi}{2} \\ &= \frac{(2n)!\pi}{2^{2n+1}(n!)^2}, \end{aligned}$$

as desired.

2. If we do the substitution $u = \frac{1}{x}$, we will have $u \to 0^+$ as $x \to \infty$, and $u \to \infty$ as $x \to 0^+$. We have $\mathrm{d}u = -\frac{1}{x^2}\,\mathrm{d}x$. Therefore,

$$\begin{aligned} J &= \int_0^\infty f((x - x^{-1})^2)\,\mathrm{d}x \\ &= \int_\infty^0 f((u^{-1} - u)^2)\left(-x^2\,\mathrm{d}u\right) \\ &= \int_0^\infty u^{-2} f((u - u^{-1})^2)\,\mathrm{d}u, \end{aligned}$$

which is exactly the first equal sign as desired (since u is just an arbitrary variable).

For the second equal sign, notice that

$$\begin{aligned} 2J &= J + J \\ &= \int_0^\infty f((x - x^{-1})^2)\,\mathrm{d}x + \int_0^\infty x^{-2} f((x - x^{-1})^2)\,\mathrm{d}x \\ &= \int_0^\infty \left(1 + x^{-2}\right) f((x - x^{-1})^2)\,\mathrm{d}x, \end{aligned}$$

and therefore

$$J = \frac{1}{2}\int_0^\infty \left(1 + x^{-2}\right) f((x - x^{-1})^2)\,\mathrm{d}x.$$

For the final equal sign, consider the substitution $u = x - x^{-1}$. Note $\mathrm{d}u = \left(1 + x^{-2}\right)\mathrm{d}x$, and when $x \to 0^+$, $u \to -\infty$, when $x \to \infty$, $u \to \infty$. Therefore,

$$\begin{aligned} J &= \frac{1}{2}\int_0^\infty \left(1 + x^{-2}\right) f((x - x^{-1})^2)\,\mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^\infty f(u^2)\,\mathrm{d}u. \end{aligned}$$

Since $f(u^2) = f((-u)^2)$ for all $u \in \mathbb{R}$, we therefore have

$$\int_{-\infty}^0 f(u^2)\,\mathrm{d}u = \int_0^\infty f(u^2)\,\mathrm{d}u,$$

and hence

$$J = \int_0^\infty f(u^2)\,\mathrm{d}u,$$

as desired.

3. Notice that the integrand satisfies

$$\begin{aligned}\frac{x^{2n-2}}{(x^4-x^2+1)^n} &= \frac{1}{x^2}\cdot\frac{(x^2)^n}{(x^4-x^2+1)^n}\\ &= \frac{1}{x^2}\cdot\frac{1}{(x^2-1+x^{-2})^n}\\ &= \frac{1}{x^2}\cdot\frac{1}{[(x-x^{-1})^2+1]^n}.\end{aligned}$$

Therefore, consider the function $f_n(x)=\frac{1}{(x+1)^n}$, we have

$$\begin{aligned}\int_0^\infty \frac{x^{2n-2}}{(x^4-x^2+1)^n}\,\mathrm{d}x &= \int_0^\infty \frac{1}{x^2}\cdot\frac{1}{[(x-x^{-1})^2+1]^n}\cdot\mathrm{d}x\\ &= \int_0^\infty x^{-2}f_n((x-x^{-1})^2)\,\mathrm{d}x\\ &= \int_0^\infty f_n(u^2)\,\mathrm{d}u\\ &= \int_0^\infty \frac{\mathrm{d}u}{(u^2+1)^n}\\ &= \frac{(2n-2)!\pi}{2^{2n-1}((n-1)!)^2}.\end{aligned}$$

2015.3 Question 2

1. Let $m = 1000$. Notice for all $n \geq m$,

$$n^2 = n \cdot n \geq m \cdot n \geq 1000n.$$

2. This statement is false. Let $s_n = (-1)^n$ and $t_n = -(-1)^n$. Then $s_n = 1$ and $t_n = -1$ for even ns, and $s_n = -1$ and $t_n = 1$ for odd ns.

   So $s_n \geq t_n$ for even ns, and $t_n \geq s_n$ for odd ns. Since there can be arbitrarily big even and odd numbers, neither of the statements are true for these sequences.

3. Let m_1 be the m for $(s_n) \leq (t_n)$ and m_2 be the m for $(t_n) \leq (u_n)$. Let $m = \max\{m_1, m_2\}$.

   Notice that for all $n \geq m$, we have $n \geq m_1$ and therefore $s_n leq t_n$, and $n \geq m_2$ amd therefore $t_n \leq u_n$.

   By the transitivity of the $\leq$ relation, we have therefore $s_n \leq u_n$, for all $n \geq m$. Therefore, this statement is true.

4. This statement is true. Let $m = 4$, we aim to prove that $2^n \geq n^2$ for all $n \geq m$.

   We first wish to prove the lemma: for all $n \geq 4$, we have $n^2 \geq 2n + 1$.

   This is equivalent to proving that $n^2 - 2n + 1 \geq 2$ for all $n \geq 4$.

   Notice that $n^2 - 2n + 1 = (n-1)^2 \geq (4-1)^2 = 9 \geq 2$ is true.

   This finishes our proof for the lemma.

   We show the original statement by mathematical induction.

   (a) **Base case.** For $n = 4$, we have $2^4 = 16 \geq 4^2 = 16$.

   (b) **Inductive step.** Assume that $2^k \geq k^2$ for some $n = k \geq 4$. We aim to show that $2^{k+1} \geq (k+1)^2$.

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &\geq 2 \cdot k^2 \\ &= k^2 + k^2 \\ &\geq k^2 + 2k + 1 \\ &= (k+1)^2. \end{aligned}$$

   Therefore, by the principle of mathematical induction, we have $2^n \geq n^2$ for all $n \geq 4$, and this finishes our proof.

2015.3 Question 3

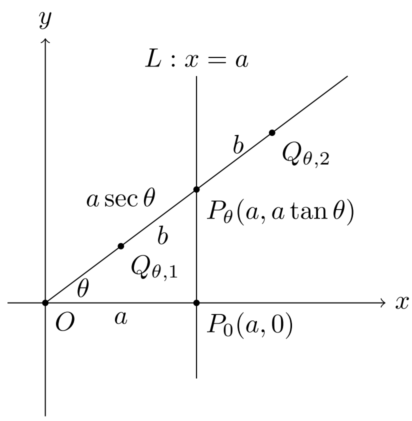
1. We prove the first part by contradiction. Assume that $\sec\theta \not> 0$, this means $\sec\theta \leq -1$.

   But in this case,
   $$r - a\sec\theta \geq r + a \geq a > b,$$
   but $|r - a\sec\theta| = b$, implies $r - a\sec\theta \leq b$, and this leads to a contradiction.

   This implies that $\sec\theta > 0$. Hence, $\cos\theta > 0$, and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

   We aim to show that $|r - a\sec\theta| = b$ lies on the conchoid of Nicomedes where $L : x = a$ and $d = b$, with $A(0,0)$.

   Let O be the origin, $P_\theta(a, a\tan\theta)$ and $P_0(a, 0)$. All points on the half-line OP_θ will have argument θ.

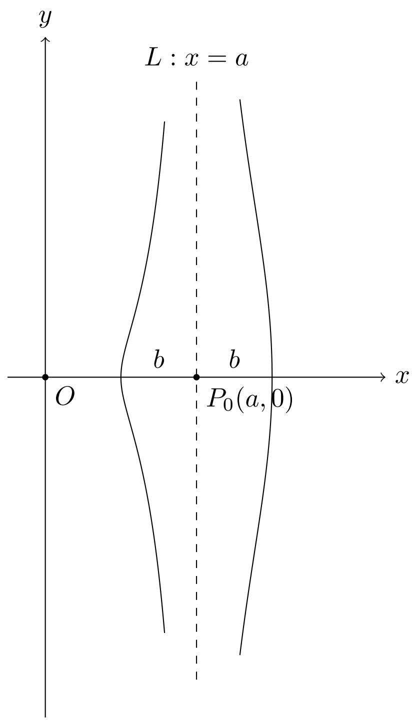


   Let Q_θ be the points on such line, satisfying the given equation $|r - a\sec\theta| = b$.

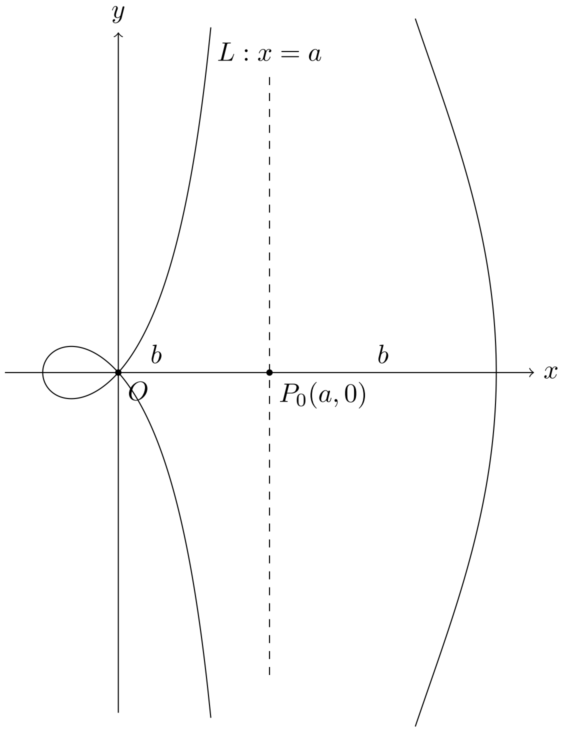
   For every $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have
   $$|OP_\theta| = |OP_0|\sec\theta = a\sec\theta.$$
   The given equation $|r - a\sec\theta| = b$ simplifies to $r = a\sec\theta \pm b$.

   This implies that Q_θ must lie on the half-line OP_θ through O, and a fixed distance b away measured along OP_θ from line $L : x = a$ (which is measured from P_θ).

   This is precisely the definition of a conchoid of Nicomedes, and this finishes our proof.



2. The sketch is as below.



When $\sec\theta < 0$, $\sec\theta \leq -1$. We have $r = a\sec\theta \pm b$.
Since $r \geq 0$, we must have $r = a\sec\theta + b \geq 0$ (since if $r = a\sec\theta - b$, then $r < 0$), and hence

$$-1 \geq \sec\theta \geq -\frac{b}{a}, -1 \leq \cos\theta \leq -\frac{a}{b},$$

which means the area of the loop is given by the range of

$$\theta \in \left(-\pi, -\arccos\left(-\frac{a}{b}\right)\right] \cup \left[\arccos\left(-\frac{a}{b}\right), \pi\right].$$

Therefore, the area of the loop is given by

$$A = \frac{1}{2}\left[\int_{-\pi}^{-\arccos\left(-\frac{a}{b}\right)} r^2 \,\mathrm{d}\theta + \int_{\arccos\left(-\frac{a}{b}\right)}^{\pi} r^2 \,\mathrm{d}\theta\right].$$

Notice that

$$\begin{aligned}\int r^2 \,\mathrm{d}\theta &= \int \left(a^2\sec^2\theta + 2ab\sec\theta + b^2\right) \mathrm{d}\theta \\ &= a^2\tan\theta + 2ab\ln|\sec\theta + \tan\theta| + b^2\theta + C \\ &= \tan\theta + 4\ln|\sec\theta + \tan\theta| + 4\theta + C,\end{aligned}$$

and

$$\alpha = \arccos\left(-\frac{a}{b}\right) = \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}.$$

Therefore,

$$\begin{aligned}
A &= \frac{1}{2}\left[\int_{-\pi}^{-\arccos\left(-\frac{a}{b}\right)} r^2 \, \mathrm{d}\theta + \int_{\arccos\left(-\frac{a}{b}\right)}^{\pi} r^2 \, \mathrm{d}\theta\right] \\
&= \frac{1}{2}\left[\left(\tan\theta + 4\ln\lvert\sec\theta + \tan\theta\rvert + 4\theta\right)_{-\pi}^{-\frac{2\pi}{3}} + \left(\tan\theta + 4\ln\lvert\sec\theta + \tan\theta\rvert + 4\theta\right)_{\frac{2\pi}{3}}^{\pi}\right] \\
&= \frac{1}{2}\left[\left(\sqrt{3} + 4\ln\left\lvert -2 + \sqrt{3}\right\rvert - \frac{8\pi}{3}\right) - \left(0 + 4\ln\lvert -1\rvert - 4\pi\right)\right. \\
&\quad \left. + \left(0 + 4\ln\lvert -1\rvert + 4\pi\right) - \left(-\sqrt{3} + 4\ln\left\lvert -2 - \sqrt{3}\right\rvert + \frac{8\pi}{3}\right)\right] \\
&= \frac{1}{2}\left(2\sqrt{3} - \frac{16\pi}{3} + 8\pi\right) + 2\ln(2 - \sqrt{3}) - 2\ln(2 + \sqrt{3}) \\
&= \frac{4}{3}\pi + \sqrt{3} + 2\ln\left(\frac{2 - \sqrt{3}}{2 + \sqrt{3}}\right) \\
&= \frac{4}{3}\pi + \sqrt{3} + 4\ln(2 - \sqrt{3}).
\end{aligned}$$

2015.3 Question 4

1. Let $f(z) = z^3 + az^2 + bz + c$. If we restrict the domain to the reals, we have

$$\lim_{x\to\infty} f(x) = \infty, \lim_{x\to-\infty} f(x) = -\infty.$$

By the definition of a limit, this means that $f(x) > 0$ for sufficiently big xs (say, for all $x \geq A$), and $f(x) < 0$ for sufficiently small xs (say, for all $x \leq B$).

Since f is continuous on $[B, A] \subset \mathbb{R}$, and $f(B) < 0$, $f(A) > 0$. This means that for some $\xi \in (B, A) \subset \mathbb{R}$ such that $f(\xi) = 0$, which finishes our proof.

2. By Vieta's Theorem, we have

$$\begin{aligned}
z_1 + z_2 + z_3 &= -a, \\
z_1z_2 + z_1z_3 + z_2z_3 &= b, \\
z_1z_2z_3 &= -c.
\end{aligned}$$

Therefore, we have $S_1 = -a$ and $a = -S_1$. Notice that

$$\begin{aligned}
\frac{S_1^2 - S_2}{2} &= \frac{(z_1 + z_2 + z_3)^2 - (z_1^2 + z_2^2 + z_3^2)}{2} \\
&= \frac{2 \cdot (z_1z_2 + z_1z_3 + z_2z_3)}{2} \\
&= z_1z_2 + z_1z_3 + z_2z_3 \\
&= b.
\end{aligned}$$

This means

$$\begin{aligned}
a &= -S_1, \\
b &= \frac{S_1^2 - S_2}{2}.
\end{aligned}$$

Also, notice that

$$\begin{aligned}
-S_1^3 + 3S_1S_2 - 2S_3 &= -(z_1 + z_2 + z_3)^3 + 3(z_1 + z_2 + z_3)(z_1^2 + z_2^2 + z_3^2) - 2(z_1^3 + z_2^3 + z_3^3) \\
&= -(z_1^3 + z_2^3 + z_3^3 + 3z_1z_2^2 + 3z_1z_3^2 + 3z_2z_1^2 + 3z_2z_3^2 + 3z_3z_1^2 + 3z_3z_2^2 + 6z_1z_2z_3) \\
&\quad + 3(z_1^3 + z_2^3 + z_3^3 + z_1z_2^2 + z_1z_3^2 + z_2z_1^2 + z_2z_3^2 + z_3z_1^2 + z_3z_2^2) \\
&\quad - 2(z_1^3 + z_2^3 + z_3^3) \\
&= -6z_1z_2z_3 \\
&= 6c,
\end{aligned}$$

as desired.

3. Consider the complex numbers $z_k = r_k \exp(i\theta_k)$ for $k = 1, 2, 3$.

This means that $z_k^n = r_k^n \exp(in\theta_k)$ by de Moivre's theorem, hence

$$\operatorname{Im} z_k^n = r_k^n \sin(n\theta_k).$$

This converts our condition to

$$\operatorname{Im} \sum_{k=1}^{3} z_k^n = 0$$

for $n = 1, 2, 3$.

Therefore, S_1, S_2, S_3 are real, and therefore, so are a, b, c.

Hence, by part (i), there must be some k such that z_k is real, which means θ_k is some multiple of 2π.

Since $\theta_k \in (-\pi, \pi)$, we must have $\theta_k = 0$ for such.

If $\theta_1 = 0$, $z_1 \in \mathbb{R}$. This therefore means that $z_1^n \in \mathbb{R}$, and hence

$$\operatorname{Im} \sum_{k=2}^{3} z_k^n = 0$$

for $n = 1, 2, 3$.

Consider the polynomial $(z - z_2)(z - z_3) = 0$, and let the expansion be $z^2 + pz + q = 0$.

By Vieta's Theorem, we have

$$\begin{aligned} z_2 + z_3 &= -p, \\ z_2 z_3 &= q. \end{aligned}$$

This therefore means that

$$\begin{aligned} p &= -(z_2 + z_3), \\ 2q &= (z_2 + z_3)^2 - (z_2^2 + z_3^2). \end{aligned}$$

If $z_2 + z_3 \in \mathbb{R}$ and $z_2^2 + z_3^2 \in \mathbb{R}$, then $p, q \in \mathbb{R}$, and z_2, z_3 are solutions to a real quadratic (polynomial).

Hence, the first case is z_2, z_3 are both real, which gives $\theta_2 = \theta_3 = 0$ since $r_k > 0$, and hence $\theta_2 = -\theta_3$.

The other case where z_2, z_3 are complex congruent to each other gives $\theta_2 = -\theta_3 + 2k\pi$ where $k \in \mathbb{Z}$ due to $r_k > 0$. But since $\theta_2, \theta_3 \in (-\pi, \pi)$, it must be the case that $\theta_2 = -\theta_3$, since the width of the interval is exactly 2π, and it is an open interval.

This finishes our proof.

2015.3 Question 5

1. - **Step 3.** Since $\sqrt{2} \in \mathbb{Q}$ is rational, there must exist positive integers $p, q \in \mathbb{N}$, such that
$$\sqrt{2} = \frac{p}{q}.$$
Therefore, $q \cdot \sqrt{2} = p \in \mathbb{Z}$, and therefore $q \in S$.
   - **Step 5.** Since $k \in S$, $k\sqrt{2}$ is a positive integer and $k \in \mathbb{N}$ is a positive integer, and hence
$$(\sqrt{2}-1) \cdot k \cdot \sqrt{2} = 2k - \sqrt{2}k$$
must be an integer, since $2k$ is an integer and $k\sqrt{2}$ is an integer. At the same time, it must be positive, since $\sqrt{2} > \sqrt{1} = 1$.
Also, $(\sqrt{2}-1) \cdot k = \sqrt{2}k - k$ is an integer due to $\sqrt{2}k$ being an integer and k being an integer, and it is positive.
So $(\sqrt{2}-1) \cdot k \in S$ as desired.
   - **Step 6.** Notice that $\sqrt{2} < \sqrt{4} = 2$, and hence $\sqrt{2} - 1 < \sqrt{4} - 1 = 1$. This means that
$$0 < (\sqrt{2}-1)k < k,$$
which implies that k is not the smallest positive integer in S, as defined in Step 4.
This leads to a contradiction, which means our initial assumption $\sqrt{2}$ is rational is not true, and hence $\sqrt{2}$ is irrational.

2. - **Only-if direction.** Since $2^{\frac{1}{3}} \in \mathbb{Q}$ is rational, we must have $\left(2^{\frac{1}{3}}\right)^2 \in \mathbb{Q}$ is rational as well, which finishes our proof.
   - **If direction.** Since $2^{\frac{2}{3}} \in \mathbb{Q}$ is rational, we must have $2^{\frac{2}{3}}/2 = 2^{-\frac{1}{3}} \in \mathbb{Q}$ is rational, which then implies $2^{\frac{1}{3}}$ is rational, which finishes our proof.

   (a) Assume that $2^{\frac{1}{3}}$ and $2^{\frac{2}{3}}$ are rational.

   (b) Define the set T to be the set of positive integers with the following property:
$$t \text{ is in } T \text{ if and only if } t2^{\frac{1}{3}} \text{ and } t2^{\frac{2}{3}} \text{ are integers,}$$
i.e.
$$T = \left\{t \in \mathbb{N} \mid t2^{\frac{1}{3}} \in \mathbb{N}, t2^{\frac{2}{3}} \in \mathbb{N}\right\}.$$

   (c) Set T contains at least one positive integer, since there must exist $a, b, c, d \in \mathbb{N}$ by Step 1 such that $2^{\frac{1}{3}} = \frac{a}{b}$ and $2^{\frac{2}{3}} = \frac{c}{d}$, and $bd \in T$.

   (d) Let k be the smallest positive integer in T.

   (e) Consider the number $\left(2^{\frac{1}{3}} - 1\right) k$.
Notice that since $k \in T$, we must have $k \in \mathbb{N}$ and $2^{\frac{1}{3}}k \in \mathbb{N}$. Hence, $\left(2^{\frac{1}{3}} - 1\right) k \in \mathbb{Z}$
Since $2 > 1$, we also have $2^{\frac{1}{3}} > 1^{\frac{1}{3}} = 1$, and hence $\left(2^{\frac{1}{3}} - 1\right) k \in \mathbb{N}$.
Also, notice that
$$\left(2^{\frac{1}{3}} - 1\right) k \cdot 2^{\frac{1}{3}} = 2^{\frac{2}{3}} \cdot k - 2^{\frac{1}{3}} \cdot k$$
and
$$\left(2^{\frac{1}{3}} - 1\right) k \cdot 2^{\frac{2}{3}} = k - 2^{\frac{2}{3}} \cdot k$$
must also both be integers since $k \in T$.
This means that $\left(2^{\frac{1}{3}} - 1\right) k \in T$.

   (f) But notice that $1 = 1^{\frac{1}{3}} < 2^{\frac{1}{3}} < 8^{\frac{1}{3}} = 2$, which means $0 < 2^{\frac{1}{3}} - 1 < 1$, and $0 < \left(2^{\frac{1}{3}} - 1\right) k < k$.
This contradicts with Step 4 where k is the smallest positive integer in T. This means our assumption that $2^{\frac{1}{3}}$ and $2^{\frac{2}{3}}$ are rational, is false.
So either of them is not rational. By the statement we proved earlier, both of them must be simultaneously rational or irrational, hence both of them must be irrational, which finishes our proof.

2015.3 Question 6

1. - **Only-if direction.** If w, z are real, then $u = w + z$ and $v = w^2 + z^2$ are real. Also,

$$\begin{aligned}2v - u^2 &= 2(w^2 + z^2) - (w + z)^2 \\ &= w^2 - 2wz + z^2 \\ &= (w - z)^2 \\ &\geq 0,\end{aligned}$$

which implies $u^2 \leq 2v$ as desired.

   - **If direction.** If $u, v \in \mathbb{R}$ and $u^2 \leq 2v$, we notice that

$$wz = \frac{u^2 - v}{2} \in \mathbb{R}.$$

Hence, w, z are solutions to the quadratic equation

$$x^2 - ux + \frac{u^2 - v}{2} = 0.$$

Notice all coefficients in this equation is real. The discriminant satisfies

$$\begin{aligned}\Delta &= (-u)^2 - 4 \cdot 1 \cdot \frac{u^2 - v}{2} \\ &= u^2 - 2(u^2 - v) \\ &= 2v - u^2 \\ &\geq 0,\end{aligned}$$

which implies both solutions must be real, i.e. w, z are real, as desired.

2. By simplification, we notice that letting $u = w + z$ and $v = w^2 + z^2$, we have

$$\begin{aligned}w^3 + z^3 &= (w + z)(w^2 + z^2) - wz(w + z) \\ &= (w + z)(w^2 + z^2) - \frac{1}{2}((w + z)^2 - (w^2 + z^2))(w + z) \\ &= uv - \frac{u(u^2 - v)}{2} \\ &= u\left(v - \frac{u^2 - v}{2}\right) \\ &= \frac{u}{2}\left(2v - (u^2 - v)\right) \\ &= \frac{u(3v - u^2)}{2}.\end{aligned}$$

This means,

$$-\lambda + \lambda u = \frac{u\left[3 \cdot \left(u^2 - \frac{2}{3}\right) - u^2\right]}{2}$$

which simplifies to

$$(u - 1)(u^2 + u - \lambda) = 0.$$

Therefore, $u_1 = 1$. The discriminant of the remaining quadratic is

$$\Delta = 1 + 4\lambda > 1 > 0,$$

since $\lambda > 0$.

Therefore, u must always be real.

The only case where there are less than 3 possible values of u, is when $u_1 = 1$ is also a solution to the quadratic.

This is precisely when $\lambda = u^2 + u = 1^2 + 1 = 2$.

Apart from this case, the two real solutions to the quadratic are distinct and must not be 1, and there are three real values of u.

Since u is always real, $u = w + z$ is always real and $v = w^2 + z^2$ is always real. However, notice that

$$2v - u^2 = 2 \cdot \left(u^2 - \frac{2}{3}\right) - u^2 = u^2 - \frac{4}{3}.$$

But when $u = 1$, $2v - u^2 < 0$, $2v < u^2$, and by part (i) at least one of w, z is not real.

2015.3 Question 7

Note that

$$\begin{aligned} D^2 x^a &= x \frac{\mathrm{d}}{\mathrm{d}x} \left(x \frac{\mathrm{d}}{\mathrm{d}x} x^a \right) \\ &= x \frac{\mathrm{d}}{\mathrm{d}x} \left(x \cdot a x^{a-1} \right) \\ &= ax \frac{\mathrm{d}}{\mathrm{d}x} x^a \\ &= ax \cdot a \cdot x^{a-1} \\ &= a^2 x^a, \end{aligned}$$

as desired.

1. Since we have that $\mathrm{d}x^a = x \cdot x^{a-1} \cdot a = ax^a$, and a is just a constant, then we must have

   $$D^n x^a = a^n x^a.$$

   If $P(x)$ is a polynomial of degree r, let

   $$P(x) = \sum_{k=0}^{r} t_k x^k.$$

   Therefore,

   $$D^n P(x) = \sum_{k=0}^{r} k^n t_k x^k.$$

   Notice that the highest degree term is $r^n t_r x^r$.

   Since $P(x)$ originally has degree $r \geq 1$, we have $r \neq 0$ and $t_r \neq 0$, and therefore this term is non-zero.

   This implies $D^n P(x)$ has degree r as well.

2. We show this by induction on n. The base case where $n = 0$ is trivially true if we define D^0 as the identity. Now, assume this is true for some $n = k < m - 1$, i.e.

   $$D^k (1-x)^m = (1-x)^{m-k} \cdot Q(x)$$

   for some polynomial Q, we aim to show this for $n = k + 1 < m$. We have

   $$\begin{aligned} D^{k+1}(1-x)^m &= D \left[(1-x)^{m-k} \cdot Q(x) \right] \\ &= x \left[-(m-k)(1-x)^{m-k-1} Q(x) + (1-x)^{m-k} Q'(x) \right] \\ &= (1-x)^{m-k-1} x \left[-(m-k) Q(x) + (1-x) Q'(x) \right], \end{aligned}$$

   which shows $D^{k+1}(1-x)^m$ is divisible by $(1-x)^{m-k-1}$ which finishes our induction step. Hence, by the principle of mathematical induction, the original statement holds for any $n < m$.

3. Notice that

   $$(1-x)^m = \sum_{r=0}^{m} \binom{m}{r} (-x)^r,$$

   and hence

   $$D^n (1-x)^m = \sum_{r=0}^{m} (-1)^r \binom{m}{r} r^n x^r.$$

   Evaluate this at $x = 1$, we can see

   $$\left[D^n (1-x)^m \right]_{x=1} = \sum_{r=0}^{m} (-1)^r \binom{m}{r} r^n 1^r = \sum_{r=0}^{m} (-1)^r \binom{m}{r} r^n.$$

But for $n < m$, $D^n(1-x)^m$ is divisible by $(1-x)^{m-n}$ and hence by $(1-x)$. This means that

$$[D^n(1-x)^m]_{x=1} = 0.$$

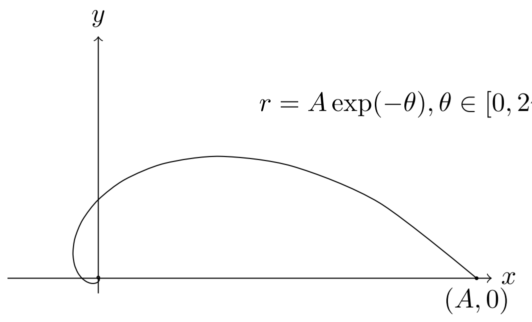
Hence,

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} r^n = 0,$$

as desired.

2015.3 Question 8

1. First, notice that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\,\mathrm{d}\theta}{\mathrm{d}x/\,\mathrm{d}\theta} = \frac{\mathrm{d}r/\,\mathrm{d}\theta \cdot \sin\theta + r \cdot \cos\theta}{\mathrm{d}r/\,\mathrm{d}\theta \cdot \cos\theta - r \cdot \sin\theta}.$$
Therefore, the original differential equation reduces to
$$(r\sin\theta + r\cos\theta)\frac{\mathrm{d}r/\,\mathrm{d}\theta \cdot \sin\theta + r \cdot \cos\theta}{\mathrm{d}r/\,\mathrm{d}\theta \cdot \cos\theta - r \cdot \sin\theta} = r\sin\theta - r\cos\theta$$
which further reduces to (since $r \neq 0$)
$$(\sin\theta + \cos\theta)\left[\frac{\mathrm{d}r}{\mathrm{d}\theta} \cdot \sin\theta + r\cos\theta\right] = (\sin\theta - \cos\theta)\left[\frac{\mathrm{d}r}{\mathrm{d}\theta} \cdot \cos\theta - r\sin\theta\right].$$
Expanding the brackets and cancelling the equivalent terms gives us
$$r\cos^2\theta + \frac{\mathrm{d}r}{\mathrm{d}\theta}\sin^2\theta = -\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos^2\theta - r\sin^2\theta,$$
which reduces to (due to the Pythagoras Theorem $\sin^2\theta + \cos^2\theta = 1$),
$$\frac{\mathrm{d}r}{\mathrm{d}\theta} + r = 0,$$
as desired.
The rearrangement (since $r \neq 0$)
$$\frac{\mathrm{d}r}{r} = -\,\mathrm{d}\theta$$
shows that the solution to this differential equation must satisfy that (since $r > 0$)
$$\ln r = -\theta + C,$$
i.e.
$$r = A\exp(-\theta),$$
where $A > 0$.
For critical values, notice that when $\theta = 0$, $r = A$, and when $\theta = 2\pi$, $r = \frac{A}{\exp 2\pi}$, and that r is decreasing with θ. The graph will look like a spiral
A sketch is shown below, for $\theta \in [0, 2\pi)$.



2. Similar to the previous part, the equation reduces to
$$\left(\sin\theta + \cos\theta - \cos\theta \cdot r^2\right)\left[\frac{\mathrm{d}r}{\mathrm{d}\theta} \cdot \sin\theta + r\cos\theta\right] = \left(\sin\theta - \cos\theta - \sin\theta \cdot r^2\right)\left[\frac{\mathrm{d}r}{\mathrm{d}\theta} \cdot \cos\theta - r\sin\theta\right],$$
and hence, by expanding brackets and eliminating terms,
$$\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin^2\theta + r\cos^2\theta - r^3\cos^2\theta = -r\sin^2\theta - \frac{\mathrm{d}r}{\mathrm{d}\theta}\cos^2\theta + r^3\sin^2\theta,$$

which then simplifies to
$$\frac{\mathrm{d}r}{\mathrm{d}\theta} + r - r^3 = 0.$$
Notice that $r = 1$ is a solution to this differential equation. Therefore, rearranging terms, we have
$$\frac{\mathrm{d}r}{r^3 - r} = \mathrm{d}\theta.$$
By partial fractions
$$\frac{1}{r^3 - r} = -\frac{1}{r} + \frac{1}{2(r+1)} + \frac{1}{2(r-1)},$$
we therefore must have
$$\left[-\frac{1}{r} + \frac{1}{2(r+1)} + \frac{1}{2(r-1)}\right] \cdot \mathrm{d}r = \mathrm{d}\theta.$$
This therefore means that
$$\frac{1}{2}\ln|r+1| + \frac{1}{2}\ln|r-1| - \ln|r| = \theta + C,$$
for some constant $C \in \mathbb{R}$.

Combining logarithms and absolute values gives us
$$\ln\left|\frac{r^2 - 1}{r^2}\right| = 2\theta + C,$$
and therefore,
$$\frac{r^2 - 1}{r^2} = \pm \exp C \cdot \exp(2\theta),$$
and this can be simplified to
$$1 - \frac{1}{r^2} = \pm \exp C \cdot \exp(2\theta),$$
and therefore
$$r^2 = \frac{1}{1 \mp \exp C \cdot \exp(2\theta)}.$$
Let $A = \mp \exp C \neq 0$, and therefore
$$r^2 = \frac{1}{1 + A\exp(2\theta)}.$$
Notice when $r = 1$, r satisfies that
$$r^2 = 1,$$
so the general solution will be
$$r^2 = \frac{1}{1 + A\exp(2\theta)}$$
for $A \in \mathbb{R}$ which this equation makes sense.

We restrict ourselves to $\theta \in [0, 2\pi)$.

Notice that, this equation makes sense for all $A \geq 0$, since the denominator is obviously non-negative.

For $A < 0$, the denominator is decreasing in θ, and we would like it to be greater than zero for some $\theta \in [0, 2\pi)$. Therefore, we would like the maximum possible value of the denominator to be greater than, that is when $\theta = 0$:
$$1 + A\exp 0 > 0,$$
which gives $A > -1$.

We consider three cases where $r > 0$, i.e.,
$$r = \frac{1}{\sqrt{1 + A\exp(2\theta)}}.$$
Notice this always passes through $\left(\frac{1}{\sqrt{1+A}}, 0\right)$.

- When $-1 < A < 0$, the curve is not defined for

$$1 + A\exp(2\theta) \leq 0,$$

and this is precisely when

$$\exp 2\theta \geq -\frac{1}{A},$$

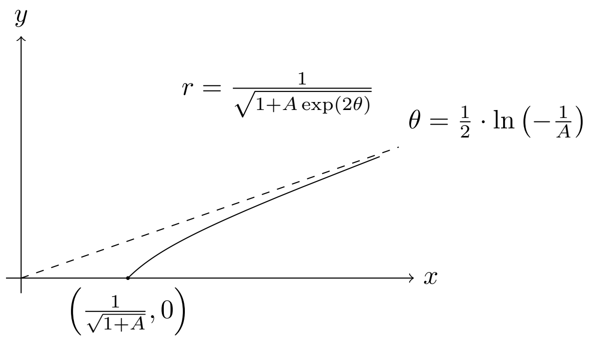
which is

$$\theta \geq \frac{1}{2} \cdot \ln\left(-\frac{1}{A}\right).$$

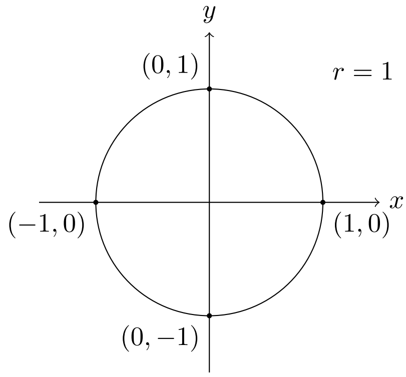
This means the curve will have an asymptote of line

$$\theta = \frac{1}{2} \cdot \ln\left(-\frac{1}{A}\right).$$

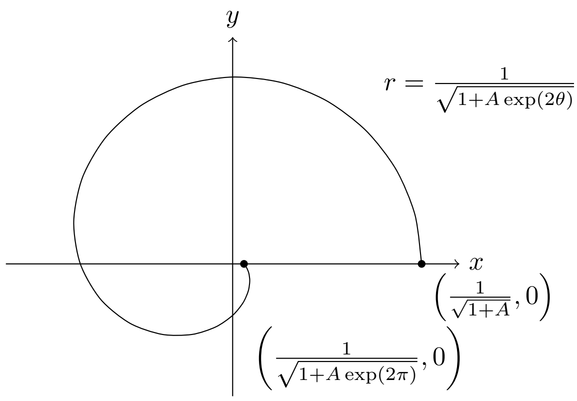
Also note that r is increasing in θ in this case, and $r \to \infty$ as $\theta \to$ the asymptote.



- When $A = 0$, notice this just gives $r = 1$, which is a circle with radius 1 centred at the origin.



- In the final case where $A > 0$, the following case arises.



2015.3 Question 12

1. Let X be the random variable for the outcome of one die roll. It has probability distribution $\mathrm{P}(X = x) = \frac{1}{6}$ for $x = 1, 2, \ldots, 6$.

   Therefore, R_1 follows the probability distribution $\mathrm{P}(R_1 = x) = \frac{1}{6}$ fir $x = 0, 1, \ldots, 5$, since $R_1 = X \bmod 6$.

   This means that
   $$G(x) = \frac{1}{6}\left(1 + x + x^2 + x^3 + x^4 + x^5\right).$$
   $R_2 = (X_1 + X_2) \bmod 6 = ((R_1)_a + (R_1)_b) \bmod 6$, and notice that,
   $$G(x)^2 = \frac{1}{36}\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 5x^6 + 4x^7 + 3x^8 + 2x^9 + x^{10}\right).$$
   Therefore, combining the terms with the same powers modulo 6, we get
   $$G_{R_2}(x) = \frac{1}{36}\left((1+5) + (2+4)x + (3+3)x^2 + (4+2)x^3 + (5+1)x^4 + 6x^5\right)$$
   which simplifies gives $G(x)$, as desired.

   Therefore, since $R_n = (X_1 + X_2 + \ldots + X_n) \bmod 6 = (R_{n-1} + R_1) \bmod 6$, by mathematical induction, we can conclude that the probability generating function for R_n is always $G(x)$.

   This means that the probability of R_n being a multiple of 6, is
   $$\mathrm{P}\left(6 \mid R_n\right) = \frac{1}{6}.$$

2. Notice that $G_1(x)$, the probability generating function for T_1 must be
   $$G_1(x) = \frac{1}{6}\left(1 + 2x + x^2 + x^3 + x^4\right).$$
   Therefore, notice that
   $$G_1(x)^2 = \frac{1}{36}\left(1 + 4x + 6x^2 + 6x^3 + 7x^4 + 6x^5 + 3x^6 + 2x^7 + x^8\right),$$
   and combining the powers with the same remainder modulo 5, we have
   $$G_2(x) = \frac{1}{36}\left(7 + 7x + 8x^2 + 7x^3 + 7x^4\right) = \frac{1}{36}\left(x^2 + 7y\right)$$
   where $y = 1 + x + x^2 + x^3 + x^4$, as desired.

   Expressing G_1 in terms of y, we have
   $$G_1(x) = \frac{1}{6}(x + y).$$
   Experimenting with G_3, we notice
   $$\begin{aligned} G_1(x) \cdot G_2(x) &= \frac{1}{6^3}(x + y)(x^2 + 7y) \\ &= \frac{1}{6^3}(x^3 + 7xy + x^2y + 7y^2). \end{aligned}$$
   But notice that up to the congruence of the powers modulo 5, we have $x^n y$ will simplify to simply y, and
   $$(x + y)^2 = x^2 + y^2 + 2xy = x^2 + 7y$$
   from $G_1(x)^2 = G_2(x)$ implies that y^2 simplifies to $5y$.

   Therefore,
   $$G_3(x) = \frac{1}{6^3}(x^3 + 7y + y + 7 \cdot 5y) = \frac{1}{6^3}(x^3 + 43y).$$

Now, we assert that

$$G_n(x) = \frac{1}{6^n}(x^{n \bmod 5} + \frac{6^n - 1}{5} y).$$

The base case is shown in G_1, and now we do the inductive step. Assume that

$$G_k(x) = \frac{1}{6^k}(x^{k \bmod 5} + \frac{6^k - 1}{5} y)$$

for some $k \in \mathbb{N}$.

$$\begin{aligned}
G_{k+1}(x) &= G_k(x) \cdot G_1(x) \\
&= \frac{1}{6^k} \cdot \left(x^{k \bmod 5} + \frac{6^k - 1}{5} y\right) \cdot \frac{1}{6} \cdot (x + y) \\
&= \frac{1}{6^{k+1}} \cdot \left(x^{k \bmod 5} \cdot x^1 + x^{k \bmod 5} \cdot y + x \cdot \frac{6^k - 1}{5} y + \frac{6^k - 1}{5} y^2\right) \\
&= \frac{1}{6^{k+1}} \cdot \left(x^{(k+1) \bmod 5} + y + \frac{6^k - 1}{5} y + \frac{6^k - 1}{5} \cdot 5y\right) \\
&= \frac{1}{6^{k+1}} \cdot \left(x^{(k+1) \bmod 5} + \left(\frac{6^k - 1}{5} + 6^k\right) y\right).
\end{aligned}$$

What remains to prove is that

$$\frac{6^k - 1}{5} + 6^k = \frac{6^{k+1} - 1}{5},$$

but this is straightforward since this is just trivial algebra.

So our assertion is true, and

$$G_n(x) = \frac{1}{6^n}(x^{n \bmod 5} + \frac{6^n - 1}{5} y).$$

Now, the probability of $5 \mid S_n$ is the coefficient of x^0 (the constant term) in $G_n(x)$.

If $5 \nmid n$, $x^{n \bmod 5}$ is not x^0, and therefore the only term that contributes to the constant term comes from y, therefore

$$\mathrm{P}\left(5 \mid S_n\right) = \frac{1}{6^n} \cdot \frac{6^n - 1}{5} = \frac{1}{5}\left(1 - \frac{1}{6^n}\right),$$

as required.

If $5 \mid n$, then $x^{n \bmod 5}$ will be $x^0 = 1$ contributing to the probability, hence

$$\mathrm{P}\left(5 \mid S_n\right) = \frac{1}{6^n} \cdot \left(1 + \frac{6^n - 1}{5}\right) = \frac{1}{5}\left(1 + \frac{4}{6^n}\right).$$

2015.3 Question 13

1. The cumulative distribution function of $X+Y$ at some value t is ratio of the area below the line $X+Y=t$ within the unit square $[0,1]^2$, against the area of the unit square (which is 1).

   When $0 \leq t \leq 1$, the area below is a triangle with vertices at $(0,0)$, $(0,t)$ and $(t,0)$. This means

   $$F_{X+Y}(t) = \frac{1}{2}t^2.$$

   When $1 \leq t \leq 2$, the area below is the unit square subtracting the triangle with vertices at $(1,1)$, $(1,t-1)$ and $(t-1,1)$. This means

   $$F_{X+Y}(t) = 1 - \frac{1}{2}[1-(t-1)]^2 = 1 - \frac{1}{2}(2-t)^2.$$

   Hence, we have

   $$F_{X+Y}(t) = \begin{cases} 0, & t < 0, \\ \frac{1}{2}t^2, & 0 \leq t < 1, \\ 1 - \frac{1}{2}(2-t)^2, & 1 \leq t < 2, \\ 1, & 2 \leq t. \end{cases}$$

2. Since $X+Y \in [0,2]$, $(X+Y)^{-1} \in \left[\frac{1}{2}, \infty\right)$. Let $t' \in \left[\frac{1}{2}, \infty\right)$, we have

   $$\begin{aligned} F_{(X+Y)^{-1}}(t') &= \mathrm{P}\left(\frac{1}{X+Y} \leq t'\right) \\ &= \mathrm{P}\left(X+Y \geq \frac{1}{t'}\right) \\ &= 1 - \mathrm{P}\left(X+Y < \frac{1}{t'}\right) \\ &= 1 - F_{X+Y}\left(\frac{1}{t'}\right). \end{aligned}$$

   If $t' \in \left[\frac{1}{2}, 1\right]$, we have $t'^{-1} \in [1,2]$, and hence

   $$\begin{aligned} F_{(X+Y)^{-1}}(t') &= 1 - F_{X+Y}\left(\frac{1}{t'}\right) \\ &= 1 - \left[1 - \frac{1}{2}\left(2 - \frac{1}{t'}\right)^2\right] \\ &= \frac{1}{2}\left(2 - \frac{1}{t'}\right)^2. \end{aligned}$$

   If $t' \in [1, \infty)$, we have $t'^{-1} \in (0,1]$, and hence

   $$\begin{aligned} F_{(X+Y)^{-1}}(t') &= 1 - F_{X+Y}\left(\frac{1}{t'}\right) \\ &= 1 - \frac{1}{2}\left(\frac{1}{t'}\right)^2. \end{aligned}$$

   Therefore, the cumulative distribution function of $(X+Y)^{-1}$ is given by

   $$F_{(X+Y)^{-1}}(t') = \begin{cases} 0, & t' < \frac{1}{2}, \\ \frac{1}{2}\left(2 - \frac{1}{t'}\right)^2, & \frac{1}{2} \leq t' < 1, \\ 1 - \frac{1}{2}\left(\frac{1}{t'}\right)^2, & 1 \leq t'. \end{cases}$$

The probability density function of $(X+Y)^{-1}$ is

$$\begin{aligned}
f_{(X+Y)^{-1}}(t') &= \frac{\mathrm{d}}{\mathrm{d}t'} F_{(X+Y)^{-1}}(t') \\
&= \begin{cases} \frac{1}{2} \cdot 2 \cdot \left(2 - \frac{1}{t'}\right) \cdot t'^{-2} = 2t'^{-2} - t'^{-3}, & \frac{1}{2} \leq t' < 1, \\ -(-2)\frac{1}{2} t'^{-3} = t'^{-3}, & 1 \leq t', \\ 0, & \text{otherwise}, \end{cases}
\end{aligned}$$

as desired.

The expectation of $\frac{1}{X+Y}$ is

$$\begin{aligned}
\mathrm{E}\left(\frac{1}{X+Y}\right) &= \int_{\mathbb{R}} t f_{(X+Y)^{-1}}(t) \, \mathrm{d}t \\
&= \int_{\frac{1}{2}}^{1} t \cdot \left(2t^{-2} - t^{-3}\right) \mathrm{d}t + \int_{1}^{\infty} t \cdot t^{-3} \, \mathrm{d}t \\
&= \int_{\frac{1}{2}}^{1} \left(2t^{-1} - t^{-2}\right) \mathrm{d}t + \int_{1}^{\infty} t^{-2} \, \mathrm{d}t \\
&= \left[2 \ln t + t^{-1}\right]_{\frac{1}{2}}^{1} - \left[t^{-1}\right]_{1}^{\infty} \\
&= \left[\left(2 \ln 1 + 1^{-1}\right) - \left(2 \ln \frac{1}{2} + \left(\frac{1}{2}\right)^{-1}\right)\right] - (0 - 1) \\
&= [1 + 2 \ln 2 - 2] + 1 \\
&= 2 \ln 2.
\end{aligned}$$

3. The cumulative distribution function of Y/X at some value t is the ratio of the area below the line $Y/X = t$ within the unit square $[0,1]^2$, against the area of the unit square.

   Since $0 \leq X, Y \leq 1$, we have $0 \leq Y/X < \infty$.

   When $0 \leq t \leq 1$, the area below is a triangle with vertices at $(0,0), (1,0)$ and $(1,t)$. Hence, we have

   $$F_{Y/X}(t) = \frac{t}{2}.$$

   When $1 \leq t < \infty$, the area below is the whole unit square, subtracting the triangle with the vertices at $(0,0), (0,1)$ and $\left(1, \frac{1}{t}\right)$. Hence, we have

   $$F_{Y/X}(t) = 1 - \frac{1}{2t}.$$

   Therefore,

   $$F_{Y/X}(t) = \begin{cases} 0, & t < 0, \\ \frac{t}{2}, & 0 \leq t < 1, \\ 1 - \frac{1}{2t}, & 1 \leq t. \end{cases}$$

   Hence, we have for $0 < t' \leq 1$, we have

   $$\begin{aligned}
   F_{\frac{X}{X+Y}}(t') &= \mathrm{P}\left(\frac{X}{X+Y} \leq t'\right) \\
   &= \mathrm{P}\left(\frac{1}{t'} \leq \frac{X+Y}{X}\right) \\
   &= \mathrm{P}\left(\frac{1}{t'} \leq 1 + \frac{Y}{X}\right) \\
   &= \mathrm{P}\left(\frac{Y}{X} \geq \frac{1}{t'} - 1\right) \\
   &= 1 - \mathrm{P}\left(\frac{Y}{X} \leq \frac{1}{t'} - 1\right) \\
   &= 1 - F_{Y/X}\left(\frac{1}{t'} - 1\right).
   \end{aligned}$$

For $0 < t' \leq \frac{1}{2}$, we have $2 \leq \frac{1}{t'}$, and hence $1 \leq \frac{1}{t'} - 1$,

$$\begin{aligned}
F_{\frac{X}{X+Y}}(t') &= 1 - F_{Y/X}\left(\frac{1}{t'} - 1\right) \\
&= 1 - \left[1 - \frac{1}{2 \cdot \left(\frac{1}{t'} - 1\right)}\right] \\
&= \frac{1}{2 \cdot \left(\frac{1}{t'} - 1\right)} \\
&= \frac{t'}{2 - 2t'}.
\end{aligned}$$

For $\frac{1}{2} \leq t' \leq 1$, we have $1 \leq \frac{1}{t'} \leq 2$, and hence $0 \leq \frac{1}{t'} \leq 1$,

$$\begin{aligned}
F_{\frac{X}{X+Y}}(t') &= 1 - F_{Y/X}\left(\frac{1}{t'} - 1\right) \\
&= 1 - \frac{\frac{1}{t'} - 1}{2} \\
&= \frac{2 - \frac{1}{t'} + 1}{2} \\
&= \frac{3t' - 1}{2t'}.
\end{aligned}$$

Hence, we have

$$F_{\frac{X}{X+Y}}(t') = \begin{cases} 0, & t' \leq 0, \\ \frac{t'}{2-2t'}, & 0 < t' \leq \frac{1}{2}, \\ \frac{3t'-1}{2t'}, & \frac{1}{2} < t' \leq 1, \\ 1, & 1 < t'. \end{cases}$$

Differentiating gives

$$\begin{aligned}
f_{\frac{X}{X+Y}}(t') &= \frac{\mathrm{d}}{\mathrm{d}t'} F_{\frac{X}{X+Y}}(t') \\
&= \begin{cases} \frac{1 \cdot (2-2t') + 2t'}{(2-2t')^2} = \frac{1}{2(1-t')^2}, & 0 < t' \leq \frac{1}{2}, \\ \frac{3 \cdot 2t' - 2(3t'-1)}{4t'^2} = \frac{1}{2t'^2}, & \frac{1}{2} < t' \leq 1, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

By symmetry, $\mathrm{E}\left(\frac{X}{X+Y}\right) = \mathrm{E}\left(\frac{Y}{X+Y}\right)$, but also

$$\mathrm{E}\left(\frac{X}{X+Y}\right) + \mathrm{E}\left(\frac{Y}{X+Y}\right) = \mathrm{E}\left(\frac{X}{X+Y} + \frac{Y}{X+Y}\right) = \mathrm{E}(1) = 1,$$

and hence

$$\mathrm{E}\left(\frac{X}{X+Y}\right) = \frac{1}{2}.$$

Using integration, we have

$$
\begin{aligned}
\mathrm{E}\left(\frac{X}{X+Y}\right) &= \int_{\mathbb{R}} x f_{\frac{X}{X+Y}}(x)\,\mathrm{d}x \\
&= \int_0^{\frac{1}{2}} \frac{x}{2(1-x)^2}\,\mathrm{d}x + \int_{\frac{1}{2}}^1 \frac{1}{2x}\,\mathrm{d}x \\
&= \int_0^{\frac{1}{2}} \frac{x}{2}\,\mathrm{d}\frac{1}{1-x} + \frac{1}{2}\left[\ln x\right]_{\frac{1}{2}}^1 \\
&= \left[\frac{x}{2(1-x)}\right]_0^{\frac{1}{2}} - \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{1-x}\,\mathrm{d}x + \frac{\ln 2}{2} \\
&= \frac{\frac{1}{2}}{2\cdot\frac{1}{2}} + \frac{1}{2}\left[\ln(1-x)\right]_0^{\frac{1}{2}} + \frac{\ln 2}{2} \\
&= \frac{1}{2} - \frac{\ln 2}{2} + \frac{\ln 2}{2} \\
&= \frac{1}{2}.
\end{aligned}
$$