Year 2014

2014 Paper 3

2014.3 Question 1

Notice that

$$(1+ax)(1+bx)(1+cx) = 1 + (a+b+c)x + (ab+ac+bc)x^2 + abcx^3,$$

and by comparing coefficients we have

$$q = bc + ca + ab, r = abc$$

1. Using the identities for the logarithms, we have

$$\begin{aligned}
\ln(1+qx^2+rx^3) &= \ln(1+ax) + \ln(1+bx) + \ln(1+cx) \\
&= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(ax)^k}{k} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(bx)^k}{k} + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(cx)^k}{k} \\
&= \sum_{k=1}^{\infty}(-1)^{k+1}x^k\frac{a^k+b^k+c^k}{k},
\end{aligned}$$

   and hence

$$S_k = \frac{a^k+b^k+c^k}{k},$$

   as desired.

2. Since

$$\begin{aligned}
S_2 &= \frac{a^2+b^2+c^2}{2} \\
&= \frac{(a+b+c)^2 - 2(ab+bc+ca)}{2} \\
&= \frac{0^2 - 2q}{2} \\
&= -q,
\end{aligned}$$

$$\begin{aligned}
S_3 &= \frac{a^3+b^3+c^3}{3} \\
&= \frac{(a+b+c)(a^2+b^2+c^2-ab-bc-ca)+3abc}{3} \\
&= abc \\
&= r,
\end{aligned}$$

   and

$$\begin{aligned}
S_5 &= \frac{a^5+b^5+c^5}{5} \\
&= \frac{(a^2+b^2+c^2)(a^3+b^3+c^3) - a^2b^2(a+b) - a^2c^2(a+c) - b^2c^2(b+c)}{5} \\
&= \frac{(-2q)(3r) + a^2b^2c + b^2c^2a + a^2c^2b}{5} \\
&= \frac{-6qr + abc(ab+bc+ac)}{5} \\
&= \frac{-6qr + qr}{5} \\
&= -qr.
\end{aligned}$$

   Therefore, $S_2S_3 = S_5$ as desired.

3. Notice that

$$\begin{aligned}
S_7 &= \frac{a^7+b^7+c^7}{7} \\
&= \frac{(a^2+b^2+c^2)(a^5+b^5+c^5)}{7} \\
&= \frac{(-2q)\cdot(-5qr) - a^2b^2(a^3+b^3) - b^2c^2(b^3+c^3) - a^2c^2(a^3+c^3)}{7} \\
&= \frac{10q^2r - a^2b^2(3r-c^3) - b^2c^2(3r-a^3) - a^2c^2(3r-b^3)}{7} \\
&= \frac{10q^2r - 3r(a^2b^2+b^2c^2+a^2c^2) + a^2b^2c^2(a+b+c)}{7} \\
&= \frac{10q^2r - 3r\left[(ab+bc+ac)^2 - 2abc(a+b+c)\right] + r^2\cdot 0}{7} \\
&= \frac{10q^2r - 3q^2r}{7} \\
&= q^2r.
\end{aligned}$$

Also, $S_2S_5 = (-q)\cdot(-qr) = q^2r$, so $S_2S_5 = S_7$ as desired.

4. Let $a=1, b=1, c=-2$. $q = bc+ca+ab = -3, r=-2$. This means $S_2 = -q = 3, S_7 = q^2r = -18$. Notice that

$$S_9 = \frac{a^9+b^9+c^9}{7} = \frac{1^9+1^9+(-2)^9}{9} = -\frac{510}{9} = -\frac{170}{3},$$

and this is obviously not S_2S_7 which gives a counterexample and the original statement is not true.

2014.3 Question 2

1. Since $u = \cosh x$, $\cosh 2x = 2\cosh^2 x - 1 = 2u^2 - 1$, and $\sinh x \, \mathrm{d}x = \mathrm{d} \cosh x = \mathrm{d}u$. Hence,

$$\begin{aligned}
\int \frac{\sinh x}{\cosh 2x} \, \mathrm{d}x &= \int \frac{\mathrm{d}u}{2u^2 - 1} \\
&= \int \frac{1}{2} \left(\frac{1}{\sqrt{2}u - 1} - \frac{1}{\sqrt{2}u + 1} \right) \mathrm{d}u \\
&= \frac{1}{2\sqrt{2}} \left(\ln\left|\sqrt{2}u - 1\right| - \ln\left|\sqrt{2}u + 1\right| \right) + C \\
&= \frac{1}{2\sqrt{2}} \ln\left| \frac{\sqrt{2}\cosh x - 1}{\sqrt{2}\cosh x + 1} \right| + C,
\end{aligned}$$

as desired.

2. Let $u = \sinh x$, $\cosh 2x = 1 + 2\sinh^2 x = 1 + 2u^2$, and $\cosh x \, \mathrm{d}x = \mathrm{d} \sinh x = \mathrm{d}u$. Hence,

$$\begin{aligned}
\int \frac{\cosh x}{\cosh 2x} \, \mathrm{d}x &= \int \frac{\mathrm{d}u}{1 + 2u^2} \\
&= \frac{1}{\sqrt{2}} \arctan()\sqrt{2}u) + C \\
&= \frac{1}{\sqrt{2}} \arctan()\sqrt{2}\sinh x) + C.
\end{aligned}$$

3. Notice that

$$\frac{\cosh x}{\cosh 2x} - \frac{\sinh x}{\cosh 2x} = \frac{2e^{-x}}{e^{2x} + e^{-2x}} = \frac{2e^x}{1 + e^{4x}}.$$

Let $u = e^x$, $\mathrm{d}u = \mathrm{d}e^x = e^x \, \mathrm{d}x$, and therefore

$$\begin{aligned}
\int_0^1 \frac{\mathrm{d}u}{1 + u^4} &= \int_{-\infty}^0 \frac{e^x \, \mathrm{d}x}{1 + x^4} \\
&= \frac{1}{2} \int_{-\infty}^0 \frac{\cosh x}{\cosh 2x} - \frac{\sinh x}{\cosh 2x} \, \mathrm{d}x \\
&= \frac{1}{2} \left[\frac{1}{\sqrt{2}} \arctan(\sqrt{2}\sinh x) - \frac{1}{2\sqrt{2}} \ln\left| \frac{\sqrt{2}\cosh x - 1}{\sqrt{2}\cosh x + 1} \right| \right]_{-\infty}^0 \\
&= \frac{1}{4\sqrt{2}} \left[2\arctan(\sqrt{2}\sinh x) - \ln\left| \frac{\sqrt{2}\cosh x - 1}{\sqrt{2}\cosh x + 1} \right| \right]_{-\infty}^0 \\
&= \frac{1}{4\sqrt{2}} \left[\left(0 - \ln\left| \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right| \right) - \left(2 \cdot (-\frac{\pi}{2}) - \ln|1| \right) \right] \\
&= \frac{1}{4\sqrt{2}} \left[\pi - 2\ln(\sqrt{2} - 1) \right] \\
&= \frac{\pi + 2\ln(\sqrt{2} + 1)}{4\sqrt{2}},
\end{aligned}$$

as desired.

2014.3 Question 3

1. Consider the point on the curve whose gradient is equal to m. Since on the curve, $x = \frac{y^2}{4a}$, and hence
$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{y}{2a} = \frac{1}{m},$$
which solves to $y_0 = \frac{2a}{m}$, and hence $x_0 = \frac{a}{m^2}$, the tangent to this point is $y = mx + \frac{a}{m}$.
If $\frac{a}{m} < c$ and $mc > a$, this means that the line $y = mx + c$ is above the tangent. Let $\theta = \arctan m$, and we know the perpendicular distance between these lines will be
$$\left(c - \frac{a}{m}\right) \cdot \cos\theta = \left(c - \frac{a}{m}\right) \cdot \frac{1}{\sqrt{m^2+1}} = \frac{cm-a}{m\sqrt{m^2+1}}.$$
If $\frac{a}{m} \geq c$ and $mc \leq a$, this means that the line $y = mx + c$ is the tangent (in the equal case) or below the tangent (in the less-than case), which both means the line $y = mx + c$ intersects with the parabola.
Hence, when $mc \leq a$, the shortest distance is always 0.

2. The distance d between $(p, 0)$ and $(at^2, 2at)$ can be expressed as
$$\begin{aligned} d^2 &= (at^2 - p)^2 + (2at)^2 \\ &= a^2t^4 - 2apt^2 + p^2 + 4a^2t^2 \\ &= a^2t^4 + 2a(2a-p)t^2 + p^2. \end{aligned}$$
We would like to minimise $d \geq 0$, which is the same as minimising d^2.
The minimum of the quadratic function
$$f(x) = a^2x^2 + 2a(2a-p)x + p^2$$
occurs when
$$x = -\frac{2a(2a-p)}{2 \cdot a^2} = \frac{p-2a}{a} = \frac{p}{a} - 2.$$
However, $d^2 = f(t^2)$ and t^2 can only be non-negative.
If $\frac{p}{a} - 2 \geq 0$, $\frac{p}{a} \geq 2$, then this value can be taken, and the minimum will be
$$d^2 = \frac{4a^2p^2 - [2a(2a-p)]^2}{4a^2} = p^2 - (2a-p)^2 = -4a^2 + 4ap = 4a(p-a)$$
and the minimal d will be
$$d = 2\sqrt{a(p-a)}.$$
In the other case where $\frac{p}{a} < 2$, to let the t^2 value to be as close as possible to the symmetric axis, we would like $t^2 = 0$, at which point the minimal distance will be
$$d^2 = f(0) = p^2,$$
and the minimal d will be
$$d = p.$$
The circle described is simply a circle centred at $(p, 0)$ with radius b. Therefore, the shortest distance will be $d - b$ if $d > b$, and 0 otherwise.
To put this into cases,
   - If $p \geq 2a$, $d = 2\sqrt{a(p-a)}$.
     - If $2\sqrt{a(p-a)} > b$, i.e. $b^2 < 4a(p-a)$, the shortest distance is $2\sqrt{a(p-a)} - b$.
     - Otherwise, $2\sqrt{a(p-a)} \leq b$, i.e. $b^2 \geq 4a(p-a)$, the shortest distance is 0.
   - Otherwise, $p < 2a$, $d = p$.
     - If $p > b$, the shortest distance is $p - b$.
     - Otherwise, $p \leq b$, the shortest distance is 0.

2014.3 Question 4

1. We have

$$\begin{aligned}
I - I_0 &= \int_0^1 \left[(y')^2 - y^2 - (y' + y\tan x)^2\right] \mathrm{d}x \\
&= -\int_0^1 \left[y^2 + y^2\tan^2 x + 2yy'\tan x\right] \mathrm{d}x \\
&= -\int_0^1 \left[y^2(1 + \tan^2 x) + 2yy'\tan x\right] \mathrm{d}x \\
&= -\int_0^1 \left(y^2 \cdot \sec^2 x + 2y \cdot y' \cdot \tan x\right) \mathrm{d}x.
\end{aligned}$$

But notice that

$$\frac{\mathrm{d}}{\mathrm{d}x} y^2 \tan x = y^2 \cdot \sec^2 x + 2y \cdot y' \cdot \tan x,$$

and hence

$$\begin{aligned}
I - I_0 &= -\int_0^1 \left(y^2 \cdot \sec^2 x + 2y \cdot y' \cdot \tan x\right) \mathrm{d}x \\
&= -\left[y^2 \tan x\right]_0^1 \\
&= -(y(1)^2 \tan 1 - 0^2 \tan 0) \\
&= -(0^2 \tan 1 - 0) \\
&= 0,
\end{aligned}$$

as desired.

This gives $I = I_1$. Also, notice that the integrand of I_1 is $(y' + y\tan x)^2$ is always non-negative, which means $I_1 \geq 0$, taking 0 only when $y' + y\tan x = 0$ for all $x \in (0, 1)$.

$$\begin{aligned}
y' + y\tan x &= 0 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -y\tan x \\
\frac{\mathrm{d}y}{y} &= -\tan x \, \mathrm{d}x \\
\ln|y| &= -\ln|\sec x| + C \\
y &= A\cos x.
\end{aligned}$$

When $x = 1$, $y = 0$, hence $A = 0$ since $\cos 1 \neq 0$. This means $I_1 = 0$ if and only if $y = 0$ for all $x \in [0, 1]$.

Since $I = I_1$, we know that $I \geq 0$, with the equal sign holding if and only if $y = 0$ for all $x \in [0, 1]$.

2. Let

$$J_0 = \int_0^1 (y' + ay\tan bx)^2 \, \mathrm{d}x,$$

and we have

$$\begin{aligned}
J - J_0 &= \int_0^1 \left[((y')^2 - a^2y^2) - (y' + ay\tan bx)^2\right] \mathrm{d}x \\
&= -\int_0^1 \left[a^2y^2 + a^2y^2\tan^2 bx + 2y' \cdot y \cdot a \cdot \tan bx\right] \mathrm{d}x \\
&= -\int_0^1 \left[a^2y^2\sec^2 bx + 2y' \cdot y \cdot a \cdot \tan bx\right] \mathrm{d}x \\
&= -a\int_0^1 \left[ay^2\sec^2 bx + 2y' \cdot y \cdot \tan bx\right] \mathrm{d}x.
\end{aligned}$$

Notice that if we let $b = a$, we have

$$\frac{\mathrm{d}y^2 \tan bx}{\mathrm{d}x} = 2yy' \tan bx + by^2 \sec^2 bx = 2yy' \tan bx + ay^2 \sec^2 bx.$$

This means

$$\begin{aligned}
J - J_0 &= -a \int_0^1 \left[ay^2 \sec^2 bx + 2y' \cdot y \cdot \tan bx\right] \mathrm{d}x \\
&= -a \left[y^2 \tan ax\right]_0^1 \\
&= -a(y(1)^2 \tan a - 0^2 \tan 0) \\
&= 0.
\end{aligned}$$

This means $J = J_0$.

Since the integrand of J_0 is a square, we know $J_0 \geq 0$ and hence $J \geq 0$.

This is only valid when $ax < \frac{\pi}{2}$ for $x \in [0, 1]$ (since otherwise this range will cross an undefined point), which means $a < \frac{\pi}{2}$.

When $a = \frac{\pi}{2}$, consider $y = \cos ax$. Notice that $y' = -a \sin ax$, and therefore

$$\begin{aligned}
J &= \int_0^1 ((-a \sin ax)^2 - a^2 \cos^2 ax) \,\mathrm{d}x \\
&= -a^2 \int_0^1 (\cos^2 ax - \sin^2 ax) \,\mathrm{d}x \\
&= -a^2 \int_0^1 \cos(2ax) \,\mathrm{d}x \\
&= -a^2 \left[\frac{\sin 2ax}{2a}\right]_0^1 \\
&= -\frac{a}{2} \left[\sin \pi x\right]_0^1 \\
&= -\frac{a}{2}(0 - 0) \\
&= 0,
\end{aligned}$$

but y is not uniformly zero.

2014.3 Question 5

$ABCD$ is a parallelogram if and only if AB is parallel and equal to DC. This is true if and only if,

$$\overrightarrow{AB} = \overrightarrow{DC},$$

and using complex representation (which is also equivalent)

$$b - a = c - d.$$

This is equivalent to

$$a + c = b + d$$

so we are done.

In this case, $ABCD$ is further a square if and only if it is both a rhombus and a rectangle. It is a rhombus if and only if the two diagonals, AC and BD, are perpendicular to each other, and a rectangle if and only if the two diagonals, AC and BD, have equal length.

This is equivalent to $\overrightarrow{BD}$ being $\overrightarrow{AC}$ rotated 90 degrees anti-clockwise exactly (due to the labelling as defined), and using complex representation (which is equivalent)

$$i(c - a) = (d - b).$$

Flipping the signs on both sides (which is reversible) gives

$$i(a - c) = (b - d)$$

as desired.

1. X is the centre of the square constructed externally along the edge PQ if and only if $\overrightarrow{PX}$ is $\overrightarrow{PQ}$ rotated clockwise by 45 degrees and scaled down by a factor of $\sqrt{2}$. In complex notation, this is equivalent to

   $$x - p = (q - p) \cdot \frac{1}{\sqrt{2}} \cdot e^{-i\frac{\pi}{4}}.$$

   But notice that $e^{-i\frac{\pi}{4}} = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1 - i)$, and hence this equation is equivalent to

   $$x = \frac{1}{2}(q - p)(1 - i) + p = \frac{(1 + i)p + (1 - i)q}{2},$$

   as desired.

2. Similarly, we have

   $$\begin{aligned} y &= \frac{(1 + i)q + (1 - i)r}{2}, \\ z &= \frac{(1 + i)r + (1 - i)s}{2}, \\ t &= \frac{(1 + i)s + (1 - i)t}{2}. \end{aligned}$$

   $XYZT$ is a square, if and only if

   $$x + z = y + t$$

   and

   $$i(x - z) = y - t.$$

   For the first one, this is equivalent to

   $$(1 + i)p + (1 - i)q + (1 + i)r + (1 - i)s = (1 - i)p + (1 + i)q + (1 - i)r + (1 + i)s,$$

   which is equivalent to

   $$p + r = q + s,$$

   which is equivalent to $PQRS$ being a parallelogram.

For the second one, this is equivalent to

$$i \cdot ((1+i)p + (1-i)q - (1+i)r - (1-i)s) = -(1-i)p + (1+i)q + (1-i)r - (1+i)s,$$

which is equivalent to

$$-(1+i)p + (1+i)q + (1-i)r - (1+i)s = -(1-i)p + (1+i)q + (1-i)r - (1+i)s,$$

which is trivially true.

This shows that $XYZT$ being square is equivalent to $PQRS$ being a parallelogram as desired.

2014.3 Question 6

Since $f''(t) > 0$ for $t \in (0, x_0)$, we must have that for all $x \in (0, x_0)$, we have $f''(t) > 0$ for $t \in (0, x)$, and hence

$$\int_0^x f''(t)\,\mathrm{d}t = f'(x) - f'(0) > 0.$$

But since $f'(0) = 0$, this implies that $f'(x) > 0$ for $x \in (0, x_0)$.

Repeating this exact step gives that $f(x) > 0$ for $x \in (0, x_0)$ as desired.

1. We would like to show $f(x) = 1 - \cos x \cosh x > 0$ for $x \in \left(0, \frac{1}{2}\pi\right)$. Notice that $f(0) = 1 - 1 \cdot 1 = 0$, and

   $$f'(x) = \sin x \cosh x - \cos x \sinh x,$$

   which means

   $$f'(0) = 0 \cdot 1 - 1 \cdot 0 = 0.$$

   Further differentiation gives

   $$f''(x) = \cos x \cosh x + \sin x \sinh x + \sin x \sinh x - \cos x \cosh x = 2 \sin x \sinh x.$$

   If $x \in \left(0, \frac{\pi}{2}\right)$, we have $\sin x > 0$ and $\sinh x > 0$, which gives $f''(x) > 0$.

   From the lemma we proved we have $f(x) > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$, which is exactly $\cos x \cosh x < 1$ as desired.

2. What is desired is to show $\sin x \cosh x - x > 0$ and $x^2 - \sin x \sinh x > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$.

   Let $g(x) = \sin x \cosh x - x$ and $h(x) = x^2 - \sin x \sinh x$. $g(0) = 0 \cdot 1 - 0 = 0$ and $h(0) = 0^2 - 0 \cdot 0 = 0$.

   Differentiating gives

   $$g'(x) = \cos x \cosh x + \sin x \sinh x - 1,$$

   and

   $$h'(x) = 2x - \cos x \sinh x - \sin x \cosh x.$$

   Hence,

   $$g'(0) = 1 \cdot 1 + 0 \cdot 0 - 1 = 0,$$

   and

   $$h'(0) = 2 \cdot 0 - 1 \cdot 0 - 0 \cdot 1 = 0.$$

   Differentiating this again gives

   $$g''(x) = -\sin x \cosh x + \cos x \sinh x + \cos x \sinh x + \sin x \cosh x = 2 \cos x \sinh x,$$

   and

   $$h''(x) = 2 + \sin x \sinh x - \cos x \cosh x - \cos x \cosh x - \sin x \sinh x = 2 - 2 \cos x \cosh x.$$

   For $x \in (0, \frac{\pi}{2})$, we notice that $\cos x > 0$ and $\sinh x > 0$, and so $g''(x) > 0$. Also, notice that $h''(x) = 2f(x)$ so $h''(x) > 0$.

   Hence, $g(x) > 0, h(x) > 0$ when $x \in (0, \frac{\pi}{2})$ which proves the result as desired.

2014.3 Question 7

1. Since P_1, P_2, P_3, P_4 are cyclic, they must satisfy that $\angle P_1P_2P_4 = \angle P_1P_3P_4$, which means $\angle P_1P_2Q = \angle QP_3P_4$. Also, we must have $\angle P_1QP_2 = \angle P_3QP_4$.

   This means that $\triangle P_1QP_2 \sim \angle P_4QP_3$ (in this order). Therefore, the ratio of the side lengths satisfy that
   $$\frac{P_1Q}{QP_2} = \frac{P_4Q}{QP_3},$$
   and hence
   $$P_1Q \cdot QP_3 = P_2Q \cdot QP_4$$
   as desired.

2. Since Q is the intersection of P_1P_3 and P_2P_4, Q is on P_1P_3, and hence the position vector of Q, $\mathbf{q}$ can be expressed as a convex combination of $\mathbf{p}_1$ and $\mathbf{p}_3$, i.e.,
   $$\mathbf{q} = b_1\mathbf{p}_1 + b_3\mathbf{p}_3$$
   where $b_1 + b_3 = 1$.

   Similarly,
   $$\mathbf{q} = b_2\mathbf{p}_2 + b_4\mathbf{p}_4$$
   where $b_2 + b_4 = 1$.

   Hence
   $$b_1\mathbf{p}_1 - b_2\mathbf{p}_2 + b_3\mathbf{p}_3 - b_4\mathbf{p}_4 = \mathbf{0}$$
   Let $a_1 = b_1, a_2 = -b_2, a_3 = b_3, a_4 = -b_4$, and we must have $\sum_{i=1}^{4} a_i = 0$, and $\sum_{i=1}^{4} a_i\mathbf{p}_i = \mathbf{0}$. Since $b_1 + b_3 = 1$ they must not be both zero, and hence a_1, a_2, a_3, a_4 are not all zero.

3. If we have $a_1 + a_3 = 0$, we must also have $a_2 + a_4 = 0$. Let $a_1 = \lambda, a_2 = \mu, a_3 = -\lambda, a_4 = -\mu$, we have
   $$\lambda(\mathbf{p}_1 - \mathbf{p}_3) = \mu(\mathbf{p}_2 - \mathbf{p}_4).$$
   But since P_1P_3 and P_2P_4 intersect at one point, this means they must not be parallel, and hence one of λ and μ must be zero. But if one of them is zero the other one has to be as well, which means all of a_i are zero, which contradicts with given.

   Still, let $b_1 = a_1, b_2 = -a_2, b_3 = a_3, b_4 = -a_4$. From given, we must have $b_1 + b_3 = b_2 + b_4 = T$. By rearrangement of the given vector equation, we have
   $$b_1\mathbf{p}_1 + b_3\mathbf{p}_3 = b_2\mathbf{p}_2 + b_4\mathbf{p}_4.$$
   If we divide both sides by T, we have
   $$\frac{b_1}{b_1 + b_3}\mathbf{p}_1 + \frac{b_3}{b_1 + b_3}\mathbf{p}_3 = \frac{b_2}{b_2 + b_4}\mathbf{p}_2 + \frac{b_4}{b_2 + b_4}\mathbf{p}_4.$$
   The position vector represented on the left-hand side must be on the line P_1P_3, and on the right-hand side must be on the line P_2P_4. But they have a unique intersection at Q, which means both must represent the position vector of Q, which is exactly
   $$\frac{a_1\mathbf{p}_1 + a_3\mathbf{p}_3}{a_1 + a_3}.$$
   It must be true that $a_3 : a_1 = P_1Q : QP_3$. This is because
   $$\mathbf{q} = \mathbf{p}_1 + \frac{a_3}{a_1 + a_3}(\mathbf{p}_3 - \mathbf{p}_1).$$
   The magnitude of $\mathbf{p}_3 - \mathbf{p}_1$ is the length P_1P_3 and the distance Q has 'travelled' along P_1P_3 from P_1 is $\frac{a_3}{a_1+a_3}$ of the total.

This means

$$P_1Q = \frac{a_3}{a_1 + a_3} P_1P_3, P_3Q = \frac{a_1}{a_1 + a_3} P_1P_3.$$

Similarly,

$$P_2Q = \frac{a_4}{a_2 + a_4} P_2P_4, P_4Q = \frac{a_2}{a_2 + a_4} P_2P_4.$$

From the first part of the question we have

$$\frac{a_1a_3}{(a_1 + a_3)^2} (P_1P_3)^2 = \frac{a_2a_4}{(a_2 + a_4)^2} (P_2P_4)^2.$$

But since $a_1 + a_2 + a_3 + a_4 = 0$, $a_1 + a_3 = -a_2 - a_4$, and hence $(a_1 + a_3)^2 = (a_2 + a_4)^2$. This means

$$a_1a_3(P_1P_3)^2 = a_2a_4(P_2P_4)^2,$$

as desired.

2014.3 Question 8

Notice that there are $(k^{n+1}-1)-k^n+1 = k^{n+1}-k^n = k^n(k-1)$ items in the summation. By the monotonic condition of the sequence in the question, we know that all the elements in the sum are greater than or equal to $f(k^n)$ and less than $f(k^{n+1})$. This immediately proves the inequality.

1. Let $k=2$. Since f is decreasing, we know that for all non-negative n, we have

$$2^n \cdot (2-1) \cdot \frac{1}{2^{n+1}} \leq \sum_{r=2^n}^{2^{n+1}-1} \frac{1}{r} \leq 2^n \cdot (2-1) \cdot \frac{1}{2^n},$$

which simplifies to

$$\frac{1}{2}1 \leq \sum_{r=2^n}^{2^{n+1}-1} \frac{1}{r} \leq 1.$$

Summing this from $n=0$ to $n=N$ (which contains $(N+1)$ such inequalities) yields

$$\frac{N+1}{2} \leq \sum_{r=1}^{2^{N+1-1}} \frac{1}{r} \leq N+1,$$

as desired.

We can show that this sum can be arbitrarily big by letting $N \to \infty$, and the lower bound of the sum $\frac{N+1}{2} \to \infty$. This means the infinite sum must diverge.

2. Let $k=2$. Since f is decreasing, we know that for all non-negative n, we have

$$\sum_{r=2^n}^{2^{n+1}-1} \frac{1}{r^3} \leq 2^n \cdot (2-1) \cdot \frac{1}{(2^n)^3} = \frac{1}{2^{2n}} = \frac{1}{4^n}.$$

Summing this from $n=0$ up to $n=N$ gives

$$\sum_{r=1}^{2^{N+1}-1} \frac{1}{r^3} \leq \sum_{n=0}^{N} \frac{1}{4^n} = \frac{1-\frac{1}{4^N}}{1-\frac{1}{4}} = \frac{4}{3} \cdot \left(1 - \frac{1}{4^{N+1}}\right).$$

Let $N \to \infty$, the weak inequality remains. This gives

$$\sum_{r=1}^{\infty} \frac{1}{r^3} \leq \frac{4}{3} \cdot 1 = \frac{4}{3}$$

as desired.

3. Using a probabilistic argument, from the set of three-digit non-negative integers (allowing leading-zeros) $\{0, 1, 2, \ldots, 999\}$, each digit has a $\frac{1}{10}$ chance of being 2, and hence $\frac{9}{10}$ chance of not being 2. This means that the number of elements in this set not being 2 is equal to

$$10^3 \cdot \left(\frac{9}{10}\right)^n = 9^3.$$

But 0 is counted in the 9^3 as well, which is not included in $S(1000)$. Therefore, $S(1000) = 9^3 - 1$.

This method applies in general to n-digit numbers and for $S(10^n) = 9^n - 1$ as well.

Let $f(i)$ be the i-th integer not having 2 in the decimal expansion in increasing order, and hence

$$S(n) = \{f(i) \mid i \in \mathbb{N}, f(i) < n\},$$

and

$$\sigma(n) = \sum_{i=1}^{S(n)} \frac{1}{f(i)}.$$

Let $k = 9$. Notice that $f(9^n) = f(S(10^n) + 1) = 10^n$ since 10^n is must be the next number satisfying the condition. Also, since f must be increasing on the integers, we have $x \mapsto \frac{1}{f(x)}$ is decreasing on the integers, and hence, for non-negative integers n

$$\sum_{r=9^n}^{9^{n+1}-1} \frac{1}{f(r)} \leq 9^n(9-1)\frac{1}{f(9^n)} = 8 \cdot \left(\frac{9}{10}\right)^n.$$

Summing this from $n = 0$ to $n = N$ gives

$$\sigma(10^{N+1}) = \sum_{r=0}^{9^{N+1}-1} \frac{1}{f(r)} \leq 8 \sum_{n=0}^{N} \left(\frac{9}{10}\right)^n = 80 \left[1 - \left(\frac{9}{10}\right)^{N+1}\right] < 80.$$

For all $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $10^{N+1} \geq n$, and since σ is increasing, we must have $80 > \sigma(10^{N+1}) \geq \sigma(n)$, which finishes the proof.

2014.3 Question 12

1. Notice that x_m is such that
$$\mathrm{P}(X \le x_m) = F(x_m) = \frac{1}{2}.$$
y_m is such that
$$\mathrm{P}(Y \le y_m) = \mathrm{P}(e^X \le y_m) = \mathrm{P}(X \le \ln y_m) = F(\ln y_m) = \frac{1}{2}.$$
Therefore,
$$F(x_m) = F(\ln y_m) = \frac{1}{2}.$$
Therefore, $x_m = \ln y_m$, and $y_m = e^{x_m}$.

2. Notice that the cumulative distribution function $G(y)$ of Y satisfies that
$$G(y) = \mathrm{P}(Y \le y) = \mathrm{P}(e^X \le y) = \mathrm{P}(X \le \ln y) = F(\ln y).$$
Therefore, differentiating both sides w.r.t. y gives that the probability density function of Y, $g(y)$ satisfies
$$g(y) = \frac{1}{y} f(\ln y),$$
as desired.
The mode of Y, λ must satisfy that $g'(\lambda) = 0$. By quotient rule, we have
$$g'(y) = \frac{f'(\ln y) \cdot \frac{1}{y} \cdot y - 1 \cdot f(\ln y)}{y^2} = \frac{f'(\ln y) - f(\ln y)}{y^2}.$$
Therefore, $g'(\lambda) = 0$ implies that $f'(\ln \lambda) = f(\ln \lambda)$ as desired.

3. This is because it is simply a horizontal shift of $f(x)$ in the positive x direction by σ^2 (i.e. this is the integral of $f(x - \sigma^2)$), and this improper integral on $\mathbb{R}$ will evaluate to the same value as integrating $f(x)$, which is simply 1.
Expanding the exponent of the integrand gives
$$\begin{aligned} -\frac{(x-\mu-\sigma^2)^2}{2\sigma^2} &= -\frac{(x-\mu)^2 + \sigma^4 - 2\sigma^2(x-\mu)}{2\sigma^2} \\ &= -\frac{(x-\mu^2)}{2\sigma^2} - \frac{1}{2}\sigma^2 + (x-\mu). \end{aligned}$$
Hence,
$$\begin{aligned} \mathrm{E}(Y) &= \mathrm{E}(e^x) \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^x \cdot e^{-(x-\mu)^2/(2\sigma^2)} \,\mathrm{d}x \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)+x} \,\mathrm{d}x \\ &= \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\mu+\frac{1}{2}\sigma^2} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)+x-\frac{1}{2}\sigma^2-\mu} \,\mathrm{d}x \\ &= e^{\mu+\frac{1}{2}\sigma^2} \cdot \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-\mu-\sigma)^2/(2\sigma^2)} \,\mathrm{d}x \\ &= e^{\mu+\frac{1}{2}\sigma^2}, \end{aligned}$$
as desired.

4. When $X \sim \mathrm{N}(\mu, \sigma^2)$, $x_m = \mu$ and therefore $y_m = e^\mu$. Differentiating the p.d.f. for X gives

$$\begin{aligned} f'(x) &= \frac{1}{\sigma\sqrt{2\pi}} \cdot \frac{-2(x-\mu)}{2\sigma^2} \cdot e^{-(x-\mu)^2/(2\sigma^2)} \\ &= -\frac{x-\mu}{\sigma^2 \cdot \sigma\sqrt{2\pi}} \cdot e^{-(x-\mu)^2/(2\sigma^2)}. \end{aligned}$$

Therefore, $f(x) = f'(x)$ when $-\frac{x-\mu}{\sigma^2} = 1$. This is precisely when $x = \mu - \sigma^2$, which means

$$\lambda = e^{\mu - \sigma^2}.$$

Now, since $\mathrm{E}(Y) = e^{\mu + \frac{1}{2}\sigma^2}, y_m = e^\mu, \lambda = e^{\mu-\sigma^2}$, and $\sigma \neq 0$ so $\sigma^2 > 0$, this gives the result

$$\lambda < y_m < \mathrm{E}(Y)$$

as desired.

2014.3 Question 13

1. Let this condition be C_1. Since the game ends in the first round, the score must remain to be zero, and therefore
$$P(N = 0 \mid C_1) = 1,$$
and for all other $n \in \mathbb{N}$ where $n \neq 0$,
$$P(N = n \mid C_1) = 0.$$
This means the p.g.f. for N conditional under C_1 is just simply $G(t \mid C_1) = P(N = 0 \mid C_1) \cdot t^0 = 1$.

2. Denote this condition be C_2. Since in the first round, the game score does not change, and after the first round it is just as if this was a new game, so for all $n \in \mathbb{N} \cup \{0\}$, we must have
$$P(N = n \mid C_2) = P(N = n),$$
and hence
$$G(t \mid C_2) = \sum_{n=0}^{\infty} P(N = n \mid C_2) \cdot x^n = \sum_{n=0}^{\infty} P(N = n) \cdot t^n = G(t).$$

3. Denote the condition where the score is increased by 1 as C_3. Since in the first round the game score increased by one, and after the first round it is just as if this was a new game, so for all $n \in \mathbb{N}$, we must have
$$P(N = n \mid C_3) = P(N = n - 1),$$
and
$$P(N = 0 \mid C_3) = 0.$$
Hence,
$$G(t \mid C_3) = \sum_{n=0}^{\infty} P(N = n \mid C_3) \cdot x^n = \sum_{n=1}^{\infty} P(N = n - 1) \cdot t^n = t \cdot \sum_{n=0}^{\infty} P(N = n) \cdot t^n = tG(t).$$
Since in the first round, one of C_1, C_2 and C_3 must happen, we must have that
$$G(t) = P(C_1) \cdot G(t \mid C_1) + P(C_2) \cdot G(t \mid C_2) + P(C_3) \cdot G(t \mid C_3) = a + bG(t) + ctG(t).$$
Hence, rearranging gives
$$(1 - b - ct)G(t) = a,$$
and hence
$$G(t) = \frac{a}{(1 - b) - ct} = \frac{a/(1 - b)}{1 - ct/(1 - b)}$$
Hence, using the infinite expansion, we have
$$\begin{aligned} G(t) &= \frac{a}{1 - b} \cdot \sum_{k=0}^{\infty} \left(\frac{ct}{1 - b}\right)^k \\ &= \sum_{k=0}^{\infty} \frac{a}{1 - b} \cdot \frac{c^k}{(1 - b)^k} \cdot t^k \\ &= \sum_{k=0}^{\infty} \frac{ac^k}{(1 - b)^{k+1}} \cdot t^k. \end{aligned}$$
But the coefficient before t^n is precisely the probability $P(N = n)$. This means
$$P(N = n) = \frac{ac^k}{(1 - b)^{k+1}},$$
as desired.

4. We know that $\mu = G'(1)$. We can find that

$$G'(t) = \frac{ac}{[(1-b) - ct]^2},$$

and evaluating this at $t = 1$ gives

$$\mu = G'(1) = \frac{ac}{(1-b-c)^2} = \frac{ac}{a^2} = \frac{c}{a}.$$

Therefore, we have $c = \mu a$

$$\begin{aligned} \mathrm{P}(N = n) &= \frac{ac^k}{(a+c)^{k+1}} \\ &= \frac{a(\mu a)^k}{(a+\mu a)^{k+1}} \\ &= \frac{a\mu^k a^k}{a^{k+1}(1+\mu)^{k+1}} \\ &= \frac{\mu^k}{\mu^{k+1}}, \end{aligned}$$

as desired.