Year 2011

2011 Paper 3

2011.3 Question 1

1. By rearrangement, we have
$$\frac{\mathrm{d}u}{u} = \left(1 + \frac{1}{x+1}\right)\mathrm{d}x,$$
and hence by integration,
$$\ln|u| = x + \ln|x+1| + C.$$
This gives
$$u = C(x+1)e^x$$
as the general solution.

2. Since $y = ze^{-x}$, we must have
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}x}e^{-x} - ze^{-x} = \left(\frac{\mathrm{d}z}{\mathrm{d}x} - z\right)e^{-x},$$
and
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}^2z}{\mathrm{d}x^2}e^{-x} - 2\frac{\mathrm{d}z}{\mathrm{d}x}e^{-x} + ze^{-x} = \left(\frac{\mathrm{d}^2z}{\mathrm{d}x^2} - 2\frac{\mathrm{d}z}{\mathrm{d}x} + z\right)e^{-x}.$$
Hence, the original differential equation can be simplified:
$$\begin{aligned}(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y &= 0\\ (x+1)\left(\frac{\mathrm{d}^2z}{\mathrm{d}x^2} - 2\frac{\mathrm{d}z}{\mathrm{d}x} + z\right)e^{-x} + x\left(\frac{\mathrm{d}z}{\mathrm{d}x} - z\right)e^{-x} - ze^{-x} &= 0\\ (x+1)\left(\frac{\mathrm{d}^2z}{\mathrm{d}x^2} - 2\frac{\mathrm{d}z}{\mathrm{d}x} + z\right) + x\left(\frac{\mathrm{d}z}{\mathrm{d}x} - z\right) - z &= 0\\ (x+1)\frac{\mathrm{d}^2z}{\mathrm{d}x^2} - (x+2)\frac{\mathrm{d}z}{\mathrm{d}x} &= 0,\end{aligned}$$
which is a first-order differential equation for $\frac{\mathrm{d}z}{\mathrm{d}x}$.
Hence, from part (i), we have the general solution to this differential equation is
$$\frac{\mathrm{d}z}{\mathrm{d}x} = C(x+1)e^x,$$
and hence by integration
$$z = C\int(x+1)e^x\,\mathrm{d}x = C\left[\int x\,\mathrm{d}e^x + \int e^x\,\mathrm{d}x\right] = C[xe^x - e^x + e^x] + D.$$
Therefore, $y = ze^{-x} = De^{-x} + Cx$. Let $A = C$ and $B = D$ and this is exactly what is desired.

3. The complementary function is the differential equation solved in the previous part. For the complementary function, consider $y = ax^2 + b$, and hence $\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2a$. Hence,
$$2a(x+1) + x\cdot 2ax - ax^2 - c = ax^2 + 2ax + (2a - c) = x^2 + 2x + 1.$$
Hence, $a = 1$ and $c = 1$ giving $y = x^2 + 1$ is a particular integral.
Therefore, the general solution to the differential equation is
$$y = Ax + Be^{-x} + x^2 + 1.$$

2011.3 Question 2

By definition,

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

where $a_n = 1$.

Hence,

$$\begin{aligned} q^{n-1} f\left(\frac{p}{q}\right) &= q^{n-1} \sum_{k=0}^{n} a_k \left(\frac{p}{q}\right)^k \\ &= q^{n-1} \sum_{k=0}^{n} a_k p^k q^{-k} \\ &= \sum_{k=0}^{n} a_k p^k q^{n-k-1}. \end{aligned}$$

For the terms with $k = 0, 1, 2, \ldots, n-1$, we have $n - k - 1 \geq 0$ and hence the terms $a_k p^k q^{n-k-1}$ is an integer, and hence the sum from $k = 0$ to $k = n - 1$ is an integer as well.

If $\frac{p}{q}$ is a rational root of f, $f\left(\frac{p}{q}\right) = 0$, and since all the rest of the terms are integers, the term where $k = n$ must be an integer as well. When $k = n$,

$$a_k p^k q^{n-k-1} = a_n p^n q^{-1} = \frac{p^n}{q}$$

must be an integer. But since p and q are co-prime, this can be an integer if and only if $q = 1$.

Therefore, $\frac{p}{q} = p$ is an integer as well, and any rational root to $f(x) = 0$ must be an integer.

1. Consider the polynomial $f(x) = x^n - 2$. The nth root of 2 must satisfy $1 < \sqrt[n]{2} < 2$, for $n \geq 2$. This is because $1^n = 1 < 2$ and $2^n = 2 \cdot 2^{n-1} > 21 = 2$.

   The nth root of 2 is a root to f. If it is rational, then it must be integer. But $1 < \sqrt[n]{2} < 2$ and so the nth root of 2 cannot be an integer. Therefore, it must be irrational.

2. Consider the polynomial $f(x) = x^3 - x + 1$. If the roots to this polynomial are rational, then they must be integer.

   Under modulo 2, $x^3 \equiv x$ since $1^3 \equiv 1$ and $0^3 \equiv 0$. Hence, $f(x) \equiv x^3 - x + 1 \equiv 0 + 1 \equiv 1$ modulo 2. This means there is no integer root to $f(x) = 0$ since the right-hand side is congruent to 0 modulo 2, and hence there are no rational roots.

3. Consider the polynomial $f(x) = x^n - 5x + 7$. If the roots to this polynomial are rational, then they must be integer.

   For $n \geq 2$, under modulo 2, $x^n \equiv 5x$ since $1^n \equiv 1 \equiv 5 \equiv 5 \cdot 1$ and $0^n \equiv 0 \equiv 5 \cdot 0$. Hence, $f(x) \equiv x^n - 5x + 7 \equiv 0 + 7 \equiv 1$ modulo 2. This means there is no integer root to $f(x) = 0$ since the right-hand side is congruent to 0 modulo 2, and hence there are no rational roots.

2011.3 Question 3

We have

$$\begin{aligned}a(x-\alpha)^3+b(x-\beta)^3&=ax^3-3a\alpha x^2+3a\alpha^2x-a\alpha^3+bx^3-3b\beta x^2+3b\beta^2x-b\beta^3\\&=(a+b)x^3-3(a\alpha+b\beta)x^2+3(a\alpha^2+b\beta^2)x-(a\alpha^3+b\beta^3).\end{aligned}$$

By comparing coefficients, we have

$$\begin{cases}a+b=1,\\-3(a\alpha+b\beta)=0\implies a\alpha+b\beta=0,\\3(a\alpha^2+b\beta^2)=-3p\implies a\alpha^2+b\beta^2=-p,\\-(a\alpha^3+b\beta^3)=q\implies a\alpha^3+b\beta^3=-q.\end{cases}$$

The first pair of equation solve to

$$(a,b)=\left(-\frac{\beta}{\alpha-\beta},\frac{\alpha}{\alpha-\beta}\right).$$

Putting this into the third equation, we can see

$$\begin{aligned}\text{LHS}&=\frac{\beta}{\beta-\alpha}\cdot\alpha^2-\frac{\alpha}{\beta-\alpha}\cdot\beta^2\\&=\frac{\alpha\beta(\alpha-\beta)}{\beta-\alpha}\\&=-\alpha\beta\\&=-\frac{p^2}{p}\\&=-p\\&=\text{RHS},\end{aligned}$$

using Vieta's Theorem for $\alpha\beta$, and for the final one,

$$\begin{aligned}\text{LHS}&=\frac{\beta}{\beta-\alpha}\cdot\alpha^3-\frac{\alpha}{\beta-\alpha}\cdot\beta^3\\&=\frac{\alpha\beta(\alpha^2-\beta^2)}{\beta-\alpha}\\&=-\frac{\alpha\beta(\alpha+\beta)(\beta-\alpha)}{\beta-\alpha}\\&=-\alpha\beta(\alpha+\beta)\\&=-\frac{p^2}{p}\cdot\left(-\frac{-q}{p}\right)\\&=-p\cdot\frac{q}{p}\\&=-q\\&=\text{RHS},\end{aligned}$$

using Vieta's Theorem for $\alpha\beta$ and $\alpha+\beta$. Hence, this means for α,β being solutions to $pt^2-qt+p^2=0$ and

$$(a,b)=\left(-\frac{\beta}{\alpha-\beta},\frac{\alpha}{\alpha-\beta}\right),$$

we have

$$x^3-3px+q=a(x-\alpha)^3+b(x-\beta)^3.$$

In this case here, we have $p=8$ and $q=48$. Hence, the quadratic equation is

$$8t^2-48t+8^2=8(t^2-6t+8)=8(t-2)(t-4)=0,$$

which solves to $(\alpha, \beta) = (2, 4)$ or $(\alpha, \beta) = (4, 2)$. Without loss of generality, let $(\alpha, \beta) = (2, 4)$, and hence

$$(a, b) = \left(-\frac{\beta}{\alpha - \beta}, \frac{\alpha}{\alpha - \beta}\right) = \left(-\frac{4}{2-4}, \frac{2}{2-4}\right) = (2, -1),$$

Hence, the original cubic equation

$$x^3 - 24x + 48 = 0$$

can be simplified to

$$2(x-2)^3 - (x-4)^3 = 0.$$

Hence,

$$2(x-2)^3 = (x-4)^3,$$

and we have

$$2^{\frac{1}{3}}(x-2) = \omega^n (x-4),$$

for $n = 0, 1, 2$ and $\omega = \exp\left(\frac{2\pi i}{3}\right)$.

Rearranging gives us

$$x = \frac{2\left(2\omega^n - 2^{\frac{1}{3}}\right)}{\omega^n - 2^{\frac{1}{3}}}$$

When $n = 0$, $\omega^n = 1$, and hence

$$x = \frac{2\left(2 - 2^{\frac{1}{3}}\right)}{1 - 2^{\frac{1}{3}}}.$$

The other two solutions

$$x = \frac{2\left(2\omega - 2^{\frac{1}{3}}\right)}{\omega - 2^{\frac{1}{3}}}, x = \frac{2\left(2\omega^2 - 2^{\frac{1}{3}}\right)}{\omega^2 - 2^{\frac{1}{3}}}.$$

This equation reduces to

$$x^3 - 3r^2x + 2r^3 = 0.$$

This can be factorised to

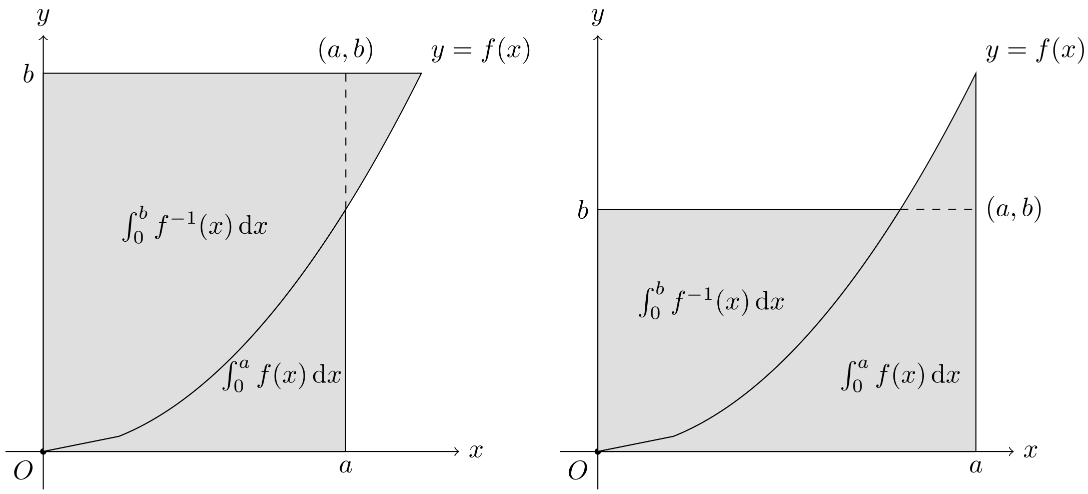
$$(x-r)(x^2 + rx - 2r^2) = (x-r)^2(x+2r)$$

and the solutions are

$$x_{1,2} = r, x_3 = -2r.$$

2011.3 Question 4

1. The following two diagrams shows the cases $a < b$ and $a > b$ respectively.



   In both cases, the shaded area is greater than the area of the rectangle formed by $(0,0),(a,0),(a,b)$ and $(0,b)$, leading to the inequality. The equal sign holds when $b = f(a)$.

2. Since $f(x) = x^{p-1}$, we must have $x = f^{-1}(x)^{p-1}$, and hence $f^{-1}(x) = x^{\frac{1}{p-1}}$. Hence,

$$\int_0^a f(x)\,\mathrm{d}x = \frac{1}{p}\left[x^p\right]_0^a = \frac{a^p}{p}.$$

   Since $\frac{1}{p} + \frac{1}{q} = 1$, we must have $\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$, and

$$q = \frac{p}{p-1},$$

   and hence

$$f^{-1}(x) = x^{q-1},$$

   which gives

$$\int_0^b f^{-1}(x)\,\mathrm{d}x = \frac{b^q}{q}.$$

   Since f is a polynomial, it must be continuous. $f(0) = 0^{p-1} = 0$, and

$$f'(x) = (p-1)x^{p-2}$$

   is always non-negative for $x \geq 0$, we must have by the original inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

   as desired.

3. Consider the function $f(x) = \sin x$. First, f is continuous, and

$$f'(x) = \cos x$$

   is always positive for $0 \leq x \leq \frac{1}{2}\pi$. We notice

$$\int_0^a f(x)\,\mathrm{d}x = \left[-\cos x\right]_0^a = 1 - \cos a,$$

and $f^{-1}(x) = \arcsin x$, and hence for $0 \leq b \leq 1$,

$$\begin{aligned}
\int_0^b f^{-1}(x)\,\mathrm{d}x &= \int_0^b \arcsin(x)\,\mathrm{d}x \\
&= [x \arcsin x]_0^b - \int_0^b x \cdot \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \\
&= \left[x \arcsin x + \sqrt{1-x^2}\right]_0^b \\
&= b \arcsin b + \sqrt{1-b^2} - 1.
\end{aligned}$$

Hence, using the given inequality,

$$ab \leq b \arcsin b + \sqrt{1-b^2} - 1 + 1 - \cos a = b \arcsin b + \sqrt{1-b^2} - \cos a,$$

as desired.

Let $a = 0$ and $b = t^{-1}$. Since $t \geq 1$, we have $0 < b \leq 1$, and hence

$$0 \leq t^{-1} \arcsin t^{-1} + \sqrt{1-t^{-2}} - \cos 0.$$

Multiplying both sides by t, and noticing $\cos 0 = 1$, we have

$$0 \leq \arcsin t^{-1} + \sqrt{t^2-1} - t,$$

and hence

$$\arcsin t^{-1} \geq t - \sqrt{t^2-1},$$

as desired.

2011.3 Question 5

Since we have

$$\tan\theta = \frac{y}{x} \implies \theta = \arctan\frac{y}{x} + k\pi$$

for some $k \in \mathbb{Z}$, differentiating with respect to t gives us

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{\mathrm{d}\frac{y}{x}}{\mathrm{d}t} = \frac{x^2}{x^2+y^2}\cdot\frac{x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}}{x^2} = \frac{x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}}{r^2}.$$

Hence,

$$\frac{1}{2}\int r^2\,\mathrm{d}\theta = \frac{1}{2}\int r^2\cdot\frac{x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}}{r^2}\cdot\mathrm{d}t = \frac{1}{2}\int\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t,$$

as desired.

The coordinates of A and B are

$$A\left(x - a\cos t, y - a\sin t\right), B\left(x + b\cos t, y + b\sin t\right).$$

Hence, we have

$$\begin{aligned}
[A] &= \frac{1}{2}\int_0^{2\pi}\left(x_A\frac{\mathrm{d}y_A}{\mathrm{d}t} - y_A\frac{\mathrm{d}x_A}{\mathrm{d}y}\right)\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi}\left[(x - a\cos t)\left(\frac{\mathrm{d}y}{\mathrm{d}t} - a\cos t\right) - (y - a\sin t)\left(\frac{\mathrm{d}x}{\mathrm{d}t} + a\sin t\right)\right]\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi}\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t - \frac{a}{2}\int_0^{2\pi}\left[\cos t\left(\frac{\mathrm{d}y}{\mathrm{d}t} + x\right) + \sin t\left(y - \frac{\mathrm{d}x}{\mathrm{d}t}\right)\right]\mathrm{d}t \\
&\quad + \frac{a^2}{2}\int_0^{2\pi}\left(\cos^2 t + \sin^2 t\right)\mathrm{d}t \\
&= [P] - af + \frac{a^2}{2}\int_0^{2\pi}\mathrm{d}t \\
&= [P] - af + 2\pi\cdot\frac{a^2}{2} \\
&= [P] + \pi a^2 - af,
\end{aligned}$$

as desired.

Similarly,

$$\begin{aligned}
[B] &= \frac{1}{2}\int_0^{2\pi}\left(x_B\frac{\mathrm{d}y_B}{\mathrm{d}t} - y_B\frac{\mathrm{d}x_B}{\mathrm{d}y}\right)\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi}\left[(x + b\cos t)\left(\frac{\mathrm{d}y}{\mathrm{d}t} + b\cos t\right) - (y + b\sin t)\left(\frac{\mathrm{d}x}{\mathrm{d}t} - b\sin t\right)\right]\mathrm{d}t \\
&= \frac{1}{2}\int_0^{2\pi}\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t + \frac{b}{2}\int_0^{2\pi}\left[\cos t\left(\frac{\mathrm{d}y}{\mathrm{d}t} + x\right) + \sin t\left(y - \frac{\mathrm{d}x}{\mathrm{d}t}\right)\right]\mathrm{d}t \\
&\quad + \frac{b^2}{2}\int_0^{2\pi}\left(\cos^2 t + \sin^2 t\right)\mathrm{d}t \\
&= [P] + bf + \frac{b^2}{2}\int_0^{2\pi}\mathrm{d}t \\
&= [P] + bf + 2\pi\cdot\frac{b^2}{2} \\
&= [P] + \pi b^2 + bf.
\end{aligned}$$

Since over $t \in [0, 2\pi]$, A and B both trace over $\mathcal{D}$, we must have

$$[A] = [B],$$

and hence

$$\pi a^2 - af = \pi b^2 + bf,$$

which means

$$\pi(a+b)(a-b) = (a+b)f,$$

and hence

$$f = (a-b)\pi,$$

and therefore

$$[A] = [B] = [P] + ab\pi.$$

The area between the curves $\mathcal{C}$ and $\mathcal{D}$ is represented as $[A] - [P]$ or $[B] - [P]$, and hence this area is πab, as desired.

2011.3 Question 6

We show that T is equal to each of U, V, X, and by transitivity, this shows that all four are equal.

- To show $T = U$, consider the substitution $u = 2\operatorname{artanh} t$, and hence $t = \tanh \frac{u}{2}$.

  When $t = \frac{1}{2}$, $u = 2\operatorname{artanh}\frac{1}{2} = 2 \cdot \frac{1}{2} \cdot \ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = \ln 3$, and when $t = \frac{1}{3}$, $u = 2\operatorname{artanh}\frac{1}{3} = 2 \cdot \frac{1}{2} \cdot \ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right) = \ln 2$.

  We have $\mathrm{d}u = \frac{2}{1-t^2}\,\mathrm{d}t$, and hence

$$\begin{aligned} T &= \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t}\,\mathrm{d}t \\ &= \int_{\ln 2}^{\ln 3} \frac{\frac{u}{2}}{\tanh\frac{u}{2}} \cdot \frac{1-\tanh^2\frac{u}{2}}{2}\,\mathrm{d}u \\ &= \int_{\ln 2}^{\ln 3} \frac{u}{2} \cdot \frac{1-\tanh^2\frac{u}{2}}{2\tanh\frac{u}{2}}\,\mathrm{d}u \\ &= \int_{\ln 2}^{\ln 3} \frac{u}{2\sinh u}\,\mathrm{d}u \\ &= U. \end{aligned}$$

- To show $T = V$, we use integration by parts.

$$\begin{aligned} T &= \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t}\,\mathrm{d}t \\ &= \int_{\frac{1}{3}}^{\frac{1}{2}} \operatorname{artanh} t\,\mathrm{d}\ln t \\ &= [\operatorname{artanh} t \ln t]_{\frac{1}{3}}^{\frac{1}{2}} - \int_{\frac{1}{3}}^{\frac{1}{2}} \ln t\,\mathrm{d}\operatorname{artanh} t \\ &= \left(\operatorname{artanh}\frac{1}{2}\ln\frac{1}{2} - \operatorname{artanh}\frac{1}{3}\ln\frac{1}{3}\right) - \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\ln t}{1-t^2}\,\mathrm{d}t \\ &= \left(\frac{1}{2}\cdot\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)\cdot(-\ln 2) - \frac{1}{2}\cdot\ln\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)\cdot(-\ln 3)\right) + V \\ &= \left(-\frac{1}{2}\cdot\ln 3\cdot\ln 2 + \frac{1}{2}\cdot\ln 2\cdot\ln 3\right) + V \\ &= V. \end{aligned}$$

- To show $T = X$, consider the substitution $x = -\frac{1}{2}\ln t$, and hence $t = e^{-2x}$.

  When $t = \frac{1}{2}$, $x = -\frac{1}{2}\ln\frac{1}{2} = \frac{1}{2}\ln 2$, and when $t = \frac{1}{3}$, $x = -\frac{1}{2}\ln\frac{1}{3} = \frac{1}{2}\ln 3$.

We have $\mathrm{d}x = -\frac{\mathrm{d}t}{2t}$, and hence

$$\begin{aligned}
T &= \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t} \,\mathrm{d}t \\
&= \int_{\frac{1}{2}\ln 3}^{\frac{1}{2}\ln 2} \frac{\operatorname{artanh} e^{-2x}}{t} \cdot (-2t) \,\mathrm{d}x \\
&= \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} 2 \operatorname{artanh} e^{-2x} \,\mathrm{d}x \\
&= \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln\left(\frac{1+e^{-2x}}{1-e^{-2x}}\right) \mathrm{d}x \\
&= \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln\left(\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}\right) \mathrm{d}x \\
&= \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln \coth x \,\mathrm{d}x \\
&= X.
\end{aligned}$$

2011.3 Question 7

1. The base case is when $n = 2$, and we have

$$T_2 = (\sqrt{a+1} + \sqrt{a})^2 = (2a+1) + 2 \cdot \sqrt{a(a+1)}.$$

We therefore have $A_2 = 2a + 1$ and $B_2 = 2$, and we verify that

$$a(a+1)B_2^2 + 1 = a(a+1) \cdot 2^2 + 1 = 4a^2 + 4a + 1 = (2a+1)^2 = A_2^2,$$

as desired, and the statement holds for the base case when $n = 2$.

Now, assume that this statement is for some even $n = k$, i.e.

$$T_k = A_k + B_k\sqrt{a(a+1)}$$

where A_k and B_k are both integers, and $A_k^2 = a(a+1)B_k^2 + 1$.

Notice that

$$\begin{aligned} T_{k+2} &= T_k \cdot \left(\sqrt{a+1} + \sqrt{a}\right)^2 \\ &= \left(A_k + B_k\sqrt{a(a+1)}\right) \cdot \left(2a+1+2\sqrt{a(a+1)}\right) \\ &= A_k \cdot (2a+1) + B_k \cdot 2 \cdot a(a+1) + 2A_k\sqrt{a(a+1)} + (2a+1)B_k\sqrt{a(a+1)} \\ &= [(2a+1)A_k + 2a(a+1)B_k] + [2A_k + (2a+1)B_k]\sqrt{a(a+1)}. \end{aligned}$$

Now let $A_{k+2} = (2a+1)A_k + 2a(a+1)B_k$, and $B_{k+2} = 2A_k + (2a+1)B_k$. Since a is a positive integer, and A_k and B_k are both integers, we must have A_{k+2} and B_{k+2} are both integers. Furthermore,

$$\begin{aligned} & A_{k+2}^2 - \left[a(a+1)B_{k+2}^2 + 1\right] \\ &= [(2a+1)A_k + 2a(a+1)B_k]^2 - \left[a(a+1)\left(2A_k + (2a+1)B_k\right)^2 + 1\right] \\ &= \left[(2a+1)^2 - 4a(a+1)\right]A_k^2 \\ &\quad + [2 \cdot (2a+1) \cdot 2a(a+1) - 2 \cdot a(a+1) \cdot 2 \cdot (2a+1)]A_kB_k \\ &\quad + [(2a(a+1))^2 - a(a+1)(2a+1)^2]B_k - 1 \\ &= A_k^2 - a(a+1)B_k^2 - 1 \\ &= 1 - 1 \\ &= 0, \end{aligned}$$

and hence

$$A_{k+2}^2 = a(a+1)B_{k+2}^2 + 1.$$

So the original statement holds for $n = k + 2$.

By the principle of mathematical induction, the original statement must hold for all even integers n.

2. If n is odd, then we have

$$\begin{aligned} T_n &= (\sqrt{a+1} + \sqrt{a})T_{n-1} \\ &= (\sqrt{a+1} + \sqrt{a})(A_{n-1} + B_{n-1}\sqrt{a(a+1)}) \\ &= A_{n-1}\sqrt{a+1} + A_{n-1}\sqrt{a} + B_{n-1}(a+1)\sqrt{a} + B_{n-1}a\sqrt{a+1} \\ &= (A_{n-1} + aB_{n-1})\sqrt{a+1} + (A_{n-1} + (a+1)B_{n-1})\sqrt{a}. \end{aligned}$$

Now, consider $C_n = A_{n-1} + aB_{n-1}$, and $D_n = A_{n-1} + (a+1)B_{n-1}$. Since a is a positive integer, and A_{n-1} and B_{n-1} are integers, we must have C_n and D_n are integers as well. Furthermore,

$$\begin{aligned}
&(a+1)C_n^2 - (aD_n^2 + 1) \\
&= (a+1)\left(A_{n-1} + aB_{n-1}\right)^2 - \left[a\left(A_{n-1} + (a+1)B_{n-1}\right)^2 + 1\right] \\
&= \left[(a+1) - a\right] A_{n-1}^2 + \left[(a+1)\cdot 2 \cdot a - a \cdot 2 \cdot (a+1)\right] A_{n-1}B_{n-1} \\
&\quad + [(a+1)a^2 - a(a+1)^2]B_{n-1}^2 - 1 \\
&= A_{n-1}^2 - a(a+1)B_{n-1}^2 - 1 \\
&= 1 - 1 \\
&= 0,
\end{aligned}$$

and hence

$$(a+1)C_n^2 = aD_n^2 + 1.$$

This shows precisely the original statement.

3. For even n,

$$T_n = A_n + B_n\sqrt{a(a+1)} = \sqrt{A_n^2} + \sqrt{B_n^2 \cdot a(a+1)} = \sqrt{A_n^2} + \sqrt{A_n^2 - 1},$$

and for odd n,

$$T_n = C_n\sqrt{a+1} + D_n\sqrt{a} = \sqrt{C_n^2(a+1)} + \sqrt{D_n^2 a} = \sqrt{aD_n^2 + 1} + \sqrt{aD_n^2},$$

as desired.

2011.3 Question 8

Since $w = u + iv$, $z = x + iy$, we have

$$\begin{aligned} u + iv &= w \\ &= \frac{1 + iz}{i + z} \\ &= \frac{1 + i(x + iy)}{i + (x + iy)} \\ &= \frac{(1 - y) + xi}{x + (y + 1)i} \\ &= \frac{(1 - y) + xi}{x + (y + 1)i} \cdot \frac{x - (y + 1)i}{x - (y + 1)i} \\ &= \frac{[(1 - y) + xi]\,[x - (y + 1)i]}{x^2 + (y + 1)^2} \\ &= \frac{(1 - y)x + x(y + 1)}{x^2 + (y + 1)^2} + \frac{x^2 - (1 - y) \cdot (y + 1)}{x^2 + (y + 1)^2} \cdot i \\ &= \frac{2x}{x^2 + (y + 1)^2} + \frac{x^2 + y^2 - 1}{x^2 + (y + 1)^2} \cdot i, \end{aligned}$$

and hence

$$(u, v) = \left(\frac{2x}{x^2 + (y + 1)^2}, \frac{x^2 + y^2 - 1}{x^2 + (y + 1)^2} \right).$$

1. When $y = 0$, we have
$$(u, v) = \left(\frac{2x}{x^2 + 1}, \frac{x^2 - 1}{x^2 + 1} \right).$$
Let $x = \tan\left(\frac{\theta}{2}\right)$. The tangent half-angle substitution also gives that $u = \sin\theta$ and $v = -\cos\theta$, and hence $u^2 + v^2 = 1$.
For the range of θ, we have $-\frac{\pi}{2} < \frac{\theta}{2} < \frac{\pi}{2}$, which means $-\pi < \theta < \pi$.
This represents the unit circle without the point $(\sin\pi, -\cos\pi) = (0, 1)$ corresponding to $\theta = \pi(+2k\pi)$ for some integer k.

2. When $-1 < x < 1$, we have $-\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{4}$, which means $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. This is the unit circle with only the part below the u axis (exclusive).

3. When $x = 0$, we have
$$(u, v) = \left(0, \frac{y^2 - 1}{(y + 1)^2} \right).$$
Notice that
$$v = \frac{y^2 - 1}{(y + 1)^2} = \frac{(y + 1)(y - 1)}{(y + 1)^2} = \frac{y - 1}{y + 1} = 1 - \frac{2}{y + 1},$$
and hence $-1 < v < 1$.
This means the locus of w is the line segment $u = 0, -1 < v < 1$.

4. When $y = 1$, we have
$$(u, v) = \left(\frac{2x}{x^2 + 4}, \frac{x^2}{x^2 + 4} \right).$$
First, let $x = 2t$, and we have
$$(u, v) = \left(\frac{4t}{4t^2 + 4}, \frac{4t^2}{4t^2 + 4} \right) = \left(\frac{t}{t^2 + 1}, \frac{t^2}{t^2 + 1} \right).$$
Let $t = \tan\left(\frac{\theta}{2}\right)$, and we have $-\pi < \theta < \pi$. Notice that
$$u = \frac{1}{2} \cdot \frac{2t}{t^2 + 1} = \frac{1}{2}\sin\theta,$$

and

$$v - \frac{1}{2} = \frac{1}{2} \cdot \frac{t^2 - 1}{t^2 + 1} = -\frac{1}{2} \cos \theta.$$

This means the loci is a subset of the circle centred at $\left(0, \frac{1}{2}\right)$ with radius $\frac{1}{2}$, with the point

$$(u, v) = \left(\frac{1}{2} \sin \pi, \frac{1}{2} - \frac{1}{2} \cos \pi\right) = (0, 1)$$

missing, which corresponds to $\theta = \pi(+2k\pi)$ for some integer k.

2011.3 Question 12

By differentiation, we have

$$[G(H(t))]' = G'(H(t)) \cdot H'(t).$$

Hence, we have

$$\begin{aligned} \mathrm{E}(Y) &= [G(H(t))]'|_{t=1} \\ &= G'(H(1)) \cdot H'(1) \\ &= G'(1) \cdot H'(1) \\ &= \mathrm{E}(N) \cdot \mathrm{E}(X_i). \end{aligned}$$

By differentiating twice, we have

$$[G(H(t))]'' = G''(H(t)) \cdot H'(t) \cdot H'(t) + G'(H(t)) \cdot H''(t).$$

Hence, we have

$$\begin{aligned} \mathrm{Var}(Y) &= \mathrm{E}(Y(Y-1)) + \mathrm{E}(Y) - \mathrm{E}(Y)^2 \\ &= [G(H(t))]''|_{t=1} + \mathrm{E}(Y) - \mathrm{E}(Y)^2 \\ &= G''(H(1)) \cdot H'(1) \cdot H'(1) + G'(H(1)) \cdot H''(1) + \mathrm{E}(Y) - \mathrm{E}(Y)^2 \\ &= G''(1) \cdot H'(1)^2 + G'(1) \cdot H''(1) + \mathrm{E}(Y) - \mathrm{E}(Y)^2 \\ &= \mathrm{E}(N(N-1)) \cdot \mathrm{E}(X_i)^2 + \mathrm{E}(N) \cdot \mathrm{E}(X_i(X_i-1)) + \mathrm{E}(Y) - \mathrm{E}(Y)^2 \\ &= \left[\mathrm{Var}(N) + \mathrm{E}(N)^2 - \mathrm{E}(N)\right] \cdot \mathrm{E}(X_i)^2 + \mathrm{E}(N) \cdot \left[\mathrm{Var}(X_i) + \mathrm{E}(X_i^2) - \mathrm{E}(X_i)\right] \\ &\quad + \mathrm{E}(N) \cdot \mathrm{E}(X_i) - \mathrm{E}(N)^2 \cdot \mathrm{E}(X_i)^2 \\ &= \mathrm{Var}(N)\,\mathrm{E}(X_i)^2 + \mathrm{E}(N)\,\mathrm{Var}(X_i). \end{aligned}$$

As defined, we have $N \sim \mathrm{Geo}\left(\frac{1}{2}\right)$, and hence

$$G(t) = \frac{\frac{1}{2} \cdot t}{1 - \left(1 - \frac{1}{2}\right) t} = \frac{t}{2-t},$$

and

$$\mathrm{E}(N) = 1/\frac{1}{2} = 2, \mathrm{Var}(N) = \frac{1 - \frac{1}{2}}{\left(\frac{1}{2}\right)^2} = 2.$$

We have $X_i \sim \mathrm{B}\left(1, \frac{1}{2}\right)$, and hence

$$H(t) = \frac{1}{2} \cdot t^0 + \frac{1}{2} \cdot t^1 = \frac{1}{2} \cdot (1+t),$$

and

$$\mathrm{E}(X_i) = 1 \cdot \frac{1}{2} = \frac{1}{2}, \mathrm{Var}(X_i) = 1 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Hence, for $Y = \sum_{i=1}^{N} X_i$, we have

$$\text{p.g.f.}_Y(t) = G(H(t)) = \frac{\frac{1}{2}(1+t)}{2 - \frac{1}{2}(1+t)} = \frac{1+t}{3-t},$$

and by the formula for expectation and variance, we have

$$\mathrm{E}(Y) = \mathrm{E}(N)\,\mathrm{E}(X_i) = 2 \cdot \frac{1}{2} = 1,$$

and

$$\mathrm{Var}(Y) = \mathrm{Var}(N) \cdot \mathrm{E}(X_i)^2 + \mathrm{E}(N) \cdot \mathrm{Var}(X_i) = 2 \cdot \left(\frac{1}{2}\right)^2 + 2 \cdot \frac{1}{4} = 1.$$

By expressing the probability generating function of Y as a power series, we notice that

$$\begin{aligned}
\text{p.g.f.}_Y(t) &= \frac{1+t}{3-t} \\
&= -1 + \frac{4}{3-t} \\
&= -1 + \frac{4}{3} \cdot \frac{1}{1-\frac{t}{3}} \\
&= -1 + \frac{4}{3} \sum_{r=0}^{\infty} \left(\frac{t}{3}\right)^r \\
&= -1 + \frac{4}{3} + \frac{4}{3} \sum_{r=1}^{\infty} 3^{-r} \cdot t^r \\
&= \frac{1}{3} + \frac{4}{3} \sum_{r=1}^{\infty} 3^{-r} \cdot t^r,
\end{aligned}$$

and hence

$$\mathrm{P}(Y = y) = \begin{cases} \frac{1}{3}, & y = 0, \\ \frac{4}{3^{y+1}}, & \text{otherwise.} \end{cases}$$

2011.3 Question 13

1. We first find the expression given by the question.

$$\begin{aligned}\frac{\mathrm{P}(X=r+1)}{\mathrm{P}(X=r)} &= \frac{\left(\frac{b}{n}\right)^{r+1}\left(\frac{n-b}{n}\right)^{k-r-1}\binom{k}{r+1}}{\left(\frac{b}{n}\right)^{r}\left(\frac{n-b}{n}\right)^{k-r}\binom{k}{r}} \\ &= \frac{b/n}{(n-b)/n}\cdot\frac{\frac{k!}{(r+1)!(k-r-1)!}}{\frac{k!}{r!(k-r)!}} \\ &= \frac{b}{n-b}\cdot\frac{r!(k-r)!}{(r+1)!(k-r-1)!} \\ &= \frac{b}{n-b}\cdot\frac{k-r}{r+1} \\ &= \frac{b}{n-b}\cdot\left(\frac{k+1}{r+1}-1\right),\end{aligned}$$

and we can see that this decreases as r increases.

If the most probable number of black balls in the sample is unique (let it be r_0), then we have

$$\mathrm{P}(X=r_0+1)<\mathrm{P}(X=r_0) \iff \frac{\mathrm{P}(X=r_0+1)}{\mathrm{P}(X=r_0)}<1,$$

and

$$\mathrm{P}(X=r_0-1)<\mathrm{P}(X=r_0) \iff \frac{\mathrm{P}(X=r_0)}{\mathrm{P}(X=r_0-1)}>1,$$

This means r_0 is the minimal solution to the inequality

$$\frac{\mathrm{P}(X=r+1)}{\mathrm{P}(X=r)}<1.$$

This could be simplified to

$$\begin{aligned}\frac{\mathrm{P}(X=r+1)}{\mathrm{P}(X=r)} &< 1 \\ \frac{b}{n-b}\left(\frac{k+1}{r+1}-1\right) &< 1 \\ \frac{k+1}{r+1}-1 &< \frac{n-b}{b} \\ \frac{k+1}{r+1} &< \frac{n}{b} \\ r+1 &> \frac{b(k+1)}{n} \\ r &> \frac{b(k+1)}{n}-1,\end{aligned}$$

and hence

$$r_0=\left\lfloor\frac{b(k+1)}{n}\right\rfloor.$$

It is not unique when there exists some r where

$$\frac{\mathrm{P}(X=r_0+1)}{\mathrm{P}(X=r_0)}=1,$$

which means there exists an integer r such that

$$r=\frac{b(k+1)}{n}-1.$$

This happens if and only if $n \mid b(k+1)$.

2. Let Y be the number of black balls in the sample. Similarly, we have

$$\begin{aligned}
\frac{\mathrm{P}(Y=r+1)}{\mathrm{P}(Y=r)} &= \frac{\frac{\binom{b}{r+1}\cdot\binom{n-b}{k-r-1}}{\binom{n}{k}}}{\frac{\binom{b}{r}\cdot\binom{n-b}{k-r}}{\binom{n}{k}}} \\
&= \frac{\frac{b!}{(r+1)!(b-r-1)!}\cdot\frac{(n-b)!}{(k-r-1)!(n+r-k-b+1)!}}{\frac{b!}{r!(b-r)!}\cdot\frac{(n-b)!}{(k-r)!(n+r-k-b)!}} \\
&= \frac{r!(b-r)!(k-r)!(n+r-k-b)!}{(r+1)!(b-r-1)!(k-r-1)!(n+r-k-b+1)!} \\
&= \frac{(b-r)\cdot(k-r)}{(r+1)\cdot(n+r-k-b+1)}.
\end{aligned}$$

The most probable number of black balls is the smallest solution to

$$\begin{aligned}
\frac{(b-r)\cdot(k-r)}{(r+1)\cdot(n+r-k-b+1)} &< 1 \\
(b-r)(k-r) &< (r+1)(n+r-k-b+1) \\
bk-rk-bk+r^2 &< nr+r^2-rk-bk+r+n+r-k-b+1 \\
(n+2)r &> bk+k+b-1-n \\
r &> \frac{bk+k+b-1-n}{n+2} \\
&= \frac{(n+1)(k+1)}{n+2}-1.
\end{aligned}$$

This means the most probable number of black balls, given its uniqueness, is

$$\left\lfloor \frac{(b+1)(k+1)}{(n+2)} \right\rfloor.$$

It is not unique when

$$\frac{(n+1)(k+1)}{n+2}-1$$

is an integer, if and only if

$$(n+2) \mid (n+1)(k+1).$$