Year 2010

2010 Paper 3

2010.3 Question 1

1. Notice that

$$\begin{aligned}C &= \frac{1}{n+1}\sum_{k=1}^{n+1} x_k \\ &= \frac{1}{n+1}\left(\sum_{k=1}^{n} x_k + x_{n+1}\right) \\ &= \frac{1}{n+1}\left(nA + x_{n+1}\right).\end{aligned}$$

2. By expanding the brackets,

$$\begin{aligned}B &= \frac{1}{n}\sum_{k=1}^{n}(x_k - A)^2 \\ &= \frac{1}{n}\sum_{k=1}^{n}(x_k^2 - 2Ax_k + A^2) \\ &= \frac{1}{n}\left[\sum_{k=1}^{n} x_k^2 - 2A\sum_{k=1}^{n} x_k + A^2 n\right] \\ &= \frac{1}{n}\sum_{k=1}^{n} x_k^2 - 2A\frac{1}{n}\sum_{k=1}^{n} x_k + A^2 \\ &= \frac{1}{n}\sum_{k=1}^{n} x_k^2 - 2A^2 + A^2 \\ &= \frac{1}{n}\sum_{k=1}^{n} x_k^2 - A^2.\end{aligned}$$

3. Similarly, we have

$$D = \frac{1}{n+1}\sum_{k=1}^{n+1} x_k^2 - C^2.$$

Hence,

$$\begin{aligned}D &= \frac{1}{n+1}\sum_{k=1}^{n+1} x_k^2 - C^2 \\ &= \frac{1}{n+1}\left(\sum_{k=1}^{n} x_k^2 + x_{n+1}^2\right) - \left(\frac{1}{n+1}(nA + x_{n+1})\right)^2 \\ &= \frac{1}{n+1}\left(n(B + A^2) + x_{n+1}^2\right) - \left(\frac{1}{n+1}(nA + x_{n+1})\right)^2 \\ &= \frac{1}{(n+1)^2}\left[(n+1)\left(n(B + A^2) + x_{n+1}^2\right) - (nA + x_{n+1})^2\right] \\ &= \frac{1}{(n+1)^2}\left(nA^2 + n(n+1)B + nx_{n+1}^2 - 2nAx_{n+1}\right) \\ &= \frac{n}{(n+1)^2}\left(A^2 + (n+1)B + x_{n+1}^2 - 2Ax_{n+1}\right) \\ &= \frac{n}{(n+1)^2}\left[(A - x_{n+1})^2 + (n+1)B\right]\end{aligned}$$

Hence,

$$\begin{aligned}(n+1)D - nB &= \frac{n}{n+1}\left[\left(A - x_{n+1}\right)^2 + (n+1)B\right] - nB \\ &= \frac{n}{n+1} \cdot \left(A - x_{n+1}\right)^2 + nB - nB \\ &= \frac{n}{n+1} \cdot \left(A - x_{n+1}\right)^2 \\ &\geq 0,\end{aligned}$$

since a square is always non-negative, and hence

$$(n+1)D \geq nB.$$

On the other hand, notice that

$$\begin{aligned}D - B &= \frac{n}{(n+1)^2}\left[\left(A - x_{n+1}\right)^2 + (n+1)B\right] - B \\ &= \frac{n}{(n+1)^2}\left(A - x_{n+1}\right)^2 + \frac{n}{n+1}B - B \\ &= \frac{n}{(n+1)^2}\left(A - x_{n+1}\right)^2 - \frac{1}{n+1}B,\end{aligned}$$

and hence

$$\begin{aligned}D < B &\iff \frac{n}{(n+1)^2}\left(A - x_{n+1}\right)^2 - \frac{1}{n+1}B < 0 \\ &\iff \frac{n}{(n+1)^2}\left(A - x_{n+1}\right)^2 < \frac{1}{n+1}B \\ &\iff \left(A - x_{n+1}\right)^2 < \frac{n+1}{n}B \\ &\iff -\sqrt{\frac{n+1}{n}B} < A - x_{n+1} < \sqrt{\frac{n+1}{n}B} \\ &\iff -A - \sqrt{\frac{n+1}{n}B} < -x_{n+1} < -A + \sqrt{\frac{n+1}{n}B} \\ &\iff A - \sqrt{\frac{n+1}{n}B} < x_{n+1} < A + \sqrt{\frac{n+1}{n}B},\end{aligned}$$

exactly as desired.

2010.3 Question 2

1. We have by definition

$$\cosh a = \frac{e^a + e^{-a}}{2}.$$

Notice that

$$\begin{aligned}\frac{1}{x^2 + 2x\cosh a + 1} &= \frac{1}{x^2 + (e^a + e^{-a})x + (e^a \cdot e^{-a})}\\ &= \frac{1}{(x + e^a)(x + e^{-a})}\\ &= \left(\frac{1}{x + e^{-a}} - \frac{1}{x + e^a}\right) \cdot \frac{1}{e^a - e^{-a}},\end{aligned}$$

and hence

$$\int \frac{\mathrm{d}x}{x^2 + 2x\cosh a + 1} = \frac{\ln|x + e^{-a}| - \ln|x + e^a|}{e^a - e^{-a}} = \frac{1}{e^a - e^{-a}} \ln\left|\frac{x + e^{-a}}{x + e^a}\right|.$$

Therefore,

$$\begin{aligned}\int_0^1 \frac{\mathrm{d}x}{x^2 + 2x\cosh a + 1} &= \frac{1}{e^a - e^{-a}}\left[\ln\left|\frac{1 + e^{-a}}{1 + e^a}\right| - \ln\left|\frac{e^{-a}}{e^a}\right|\right]\\ &= \frac{1}{e^a - e^{-a}}\left[\ln\left|\frac{1 + e^{-a}}{e^a\left(1 + e^{-a}\right)}\right| + 2a\right]\\ &= \frac{1}{e^a - e^{-a}}\left[-a + 2a\right]\\ &= \frac{a}{e^a - e^{-a}}\\ &= \frac{a}{2\sinh a}.\end{aligned}$$

2. For the first integral, we have by definition

$$\sinh a = \frac{e^a - e^{-a}}{2}.$$

Notice that

$$\begin{aligned}\frac{1}{x^2 + 2x\sinh a - 1} &= \frac{1}{x^2 + (e^a - e^{-a})x - (e^a \cdot e^{-a})}\\ &= \frac{1}{(x + e^a)(x - e^{-a})}\\ &= \left(\frac{1}{x - e^{-a}} - \frac{1}{x + e^a}\right) \cdot \frac{1}{e^a + e^{-a}},\end{aligned}$$

and hence

$$\int \frac{\mathrm{d}x}{x^2 + 2x\sinh a - 1} = \frac{\ln|x - e^{-a}| - \ln|x + e^a|}{e^a + e^{-a}} = \frac{1}{e^a + e^{-a}} \ln\left|\frac{x - e^{-a}}{x + e^a}\right|.$$

Therefore,

$$\begin{aligned}
\int_1^\infty \frac{\mathrm{d}x}{x^2+2x\sinh a-1} &= \frac{1}{e^a+e^{-a}}\cdot\left[\ln\left|\frac{x-e^{-a}}{x+e^a}\right|\right]_1^\infty \\
&= \frac{1}{2\cosh a}\cdot\left[\ln 1-\ln\left|\frac{1-e^{-a}}{1+e^a}\right|\right] \\
&= \frac{1}{2\cosh a}\cdot\ln\frac{1+e^a}{1-e^{-a}} \\
&= \frac{1}{2\cosh a}\cdot\left(a+\ln\frac{1+e^{-a}}{1-e^{-a}}\right) \\
&= \frac{1}{2\cosh a}\cdot\left(a+\ln\frac{e^{\frac{a}{2}}+e^{-\frac{a}{2}}}{e^{\frac{a}{2}}-e^{-\frac{a}{2}}}\right) \\
&= \frac{1}{2\cosh a}\cdot\left(a+\ln\coth\frac{a}{2}\right).
\end{aligned}$$

For the second integral, notice that

$$\frac{1}{x^4+2x^2\cosh a+1} = \frac{1}{e^a-e^{-a}}\left(\frac{1}{x^2+e^{-a}}-\frac{1}{x^2+e^a}\right),$$

and hence

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{d}x}{x^4+2x^2\cosh a+1} &= \frac{1}{2\sinh a}\int_0^\infty\left(\frac{1}{x^2+e^{-a}}-\frac{1}{x^2+e^a}\right)\mathrm{d}x \\
&= \frac{1}{2\sinh a}\left[e^{\frac{a}{2}}\arctan\left(e^{\frac{a}{2}}x\right)-e^{-\frac{a}{2}}\arctan\left(e^{-\frac{a}{2}}x\right)\right]_0^\infty \\
&= \frac{1}{2\sinh a}\left[\left(e^{\frac{a}{2}}\frac{\pi}{2}-e^{-\frac{a}{2}}\frac{\pi}{2}\right)-\left(e^{\frac{a}{2}}0-e^{-\frac{a}{2}}0\right)\right] \\
&= \frac{1}{2\sinh a}\cdot(e^{\frac{a}{2}}-e^{-\frac{a}{2}})\cdot\frac{\pi}{2} \\
&= \frac{\pi\sinh\frac{a}{2}}{2\sinh a} \\
&= \frac{\pi\sinh\frac{a}{2}}{4\sinh\frac{a}{2}\cosh\frac{a}{2}} \\
&= \frac{\pi}{4\cosh\frac{a}{2}}.
\end{aligned}$$

2010.3 Question 3

An n-th root of unity takes the form $\exp(\frac{k}{n} \cdot 2\pi I)$ for $k = 0, \ldots, n-1$, and specially, it is a primitive nth root of unity, if and only if the fraction $\frac{k}{n}$ is irreducible (being reducible is equivalent to it being another mth root of unity where $0 < m < n$), and this is equivalent to $\gcd(k, n) = 1$.

The two primitive 4th roots of unity are when $k = 1$ or 3, which gives i and $-i$ as the two primitive roots.

Hence,

$$C_4(x) = (x - i)(x + i) = x^2 + 1.$$

1. For $n = 1$, $k = 0$, and $\gcd(0, 1) = 1$. So the only 1st root of unity is primitive, and hence

$$C_1(x) = x - 1.$$

For $n = 2$, $k = 0$ or 1, and only $\gcd(1, 2) = 1$. So the only primitive 2nd root of unity is $\exp\left(\frac{1}{2} \cdot 2\pi i\right) = -1$, and hence

$$C_2(x) = x + 1.$$

For $n = 3$, $k = 1$ or 2 gives $\gcd(k, n) = 1$. Hence, the primitive 3rd roots of unity are all 3rd roots of unity apart from $x = 1$. Hence,

$$C_3(x) = \frac{x^3 - 1}{x - 1} = x^2 + x + 1.$$

For $n = 5$, $k = 1, 2, 3, 4$ or 5 gives $\gcd(k, n) = 1$. Hence, the primitive 5th roots of unity are all 5th roots of unity apart from $x = 1$. Hence,

$$C_5(x) = \frac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1.$$

For $n = 6$, $k = 1$ or 5 gives $\gcd(k, n) = 1$. Hence,

$$\begin{aligned} C_6(x) &= \left(x - \exp\left(\frac{1}{6} \cdot 2\pi i\right)\right)\left(x - \exp\left(\frac{5}{6} \cdot 2\pi i\right)\right) \\ &= \left(x - \exp\left(\frac{1}{3} \cdot \pi i\right)\right)\left(x - \exp\left(-\frac{1}{3} \cdot \pi i\right)\right) \\ &= x^2 - 2 \cdot \cos\left(\frac{1}{3} \cdot \pi\right) x + 1 \\ &= x^2 - x + 1. \end{aligned}$$

2. Notice that

$$\begin{aligned} x^4 + 1 &= (x^2 + i)(x^2 - i) \\ &= \left[x^2 - \exp\left(\frac{3}{4} \cdot 2\pi i\right)\right]\left[x^2 - \exp\left(\frac{1}{4} \cdot 2\pi i\right)\right] \\ &= \left[x - \exp\left(\frac{3}{8} \cdot 2\pi i\right)\right]\left[x - \exp\left(\frac{7}{8} \cdot 2\pi i\right)\right]\left[x - \exp\left(\frac{1}{8} \cdot 2\pi i\right)\right]\left[x - \exp\left(\frac{5}{8} \cdot 2\pi i\right)\right], \end{aligned}$$

and the roots to $C_n(x)$ are

$$\exp\left(\frac{1}{8} \cdot 2\pi i\right), \exp\left(\frac{3}{8} \cdot 2\pi i\right), \exp\left(\frac{5}{8} \cdot 2\pi i\right), \exp\left(\frac{7}{8} \cdot 2\pi i\right).$$

Since the number on the denominator is 8 (and all fractions are reduced), we can conclude that if n exists, then $n = 8$.

On the other hand, for $n = 8$, only $k = 1, 3, 5$ and 7 give $\gcd(k, n) = 1$. This means that $n = 8$ satisfies that the primitive 8-th roots of unity being

$$\exp\left(\frac{1}{8} \cdot 2\pi i\right), \exp\left(\frac{3}{8} \cdot 2\pi i\right), \exp\left(\frac{5}{8} \cdot 2\pi i\right), \exp\left(\frac{7}{8} \cdot 2\pi i\right).$$

Hence, $n = 8$ satisfies $C_n(x) = x^4 + 1$, and hence $n = 8$.

3. Since p is prime, for $k = 1, 2, 3, \ldots, p-1$, we must have $\gcd(k, p) = 1$ (and for $k = 0$, $\gcd(k, p) = p \neq 1$). This means that all the pth roots of unity apart from $x = 1$ will be primitive pth roots of unity, and hence
$$C_p(x) = \frac{x^p - 1}{x - 1} = 1 + x + x^2 + \cdots + x^{p-1}.$$

4. A root of C_q must take the form of
$$\exp\left(\frac{Q}{q} \cdot 2\pi i\right)$$
where $0 \leq Q < q, \gcd(Q, q) = 1$.

A root of C_r must take the form of
$$\exp\left(\frac{R}{r} \cdot 2\pi i\right)$$
where $0 \leq R < r, \gcd(R, r) = 1$, and a root of C_s must take the form of
$$\exp\left(\frac{S}{s} \cdot 2\pi i\right)$$
where $0 \leq S < s, \gcd(S, s) = 1$.

Since a root to C_s must be a root to the right-hand side of the equation, and hence must be a root to the left-hand side of the equation, we have
$$\exp\left(\frac{Q}{q} \cdot 2\pi i\right) = \exp\left(\frac{S}{s} \cdot 2\pi i\right).$$

Since $0 \leq \frac{Q}{q}, \frac{S}{s} < 1$, we must have
$$\frac{Q}{q} = \frac{S}{s},$$
and since they are both reduced fractions, we must have $q = s$.

Similarly, we also have $q = r$.

This means
$$C_q(x) = C_q(x)^2,$$
and hence
$$C_q(x)(C_q(x) - 1) = 0.$$

Since C_q is a polynomial, this means either $C_q(x) = 0$ or $C_q(x) = 1$, both of which are not possible given q is a positive integer. For the first case, this is impossible since this polynomial has infinitely many roots, but there are only finitely many qth roots of unity, and hence only finitely many primitive qth roots of unity.

For the second case, this means that there is no primitive qth roots of unity. But for $k = 1$, $\gcd(k, q) = 1$, and hence there must be a primitive qth root of unity
$$\exp\left(\frac{1}{q} \cdot 2\pi i\right),$$
and this must be impossible.

Hence, there are no positive integers q, r and s such that
$$C_q(x) = C_r(x) \cdot C_s(x).$$

2010.3 Question 4

1. Since α is a common root of both equations, we have

$$\alpha^2 + a\alpha + b = 0, \alpha^2 + c\alpha + d = 0.$$

Since $0 = 0$, we have

$$\begin{aligned} \alpha^2 + a\alpha + b &= \alpha^2 + c\alpha + d \\ a\alpha + b &= c\alpha + d \\ (a-c)\alpha &= -(b-d) \\ \alpha &= -\frac{b-d}{a-c}, \end{aligned}$$

given that $a \neq c$.

We first prove the only-if direction of the statement. Putting this back to the first original equation, we have

$$\left(-\frac{b-d}{a-c}\right)^2 + a \cdot \left(-\frac{b-d}{a-c}\right) + b = 0,$$

and hence multiplying both sides by $(a-c)^2$, we get

$$(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0,$$

as desired.

The if direction of the statement is as follows. Given this equation, dividing both sides by $(a-c)^2$ gives

$$\left(-\frac{b-d}{a-c}\right)^2 + a\left(-\frac{b-d}{a-c}\right) + b = 0,$$

and putting $x = -\frac{b-d}{a-c}$ into the second equation gives

$$\begin{aligned} x^2 + cx + d &= \left(-\frac{b-d}{a-c}\right)^2 + c\left(-\frac{b-d}{a-c}\right) + d \\ &= \frac{1}{(a-c)^2}\left[(b-d)^2 - c(b-d)(a-c) + d(a-c)^2\right] \\ &= \frac{1}{(a-c)^2}\left[(b-d)^2 - a(b-d)(a-c) + a(b-d)(a-c) - c(b-d)(a-c) + d(a-c)^2\right] \\ &= \frac{1}{(a-c)^2}\left[(b-d)^2 - a(b-d)(a-c) + (b-d)(a-c)^2 + d(a-c)^2\right] \\ &= \frac{1}{(a-c)^2}\left[(b-d)^2 - a(b-d)(a-c) + b(a-c)^2\right] \\ &= 0. \end{aligned}$$

This still holds if $a \neq c$. For the only-if direction, we still have $(a-c)\alpha = -(b-d)$, and hence $(a-c)^2\alpha^2 = (b-d)^2$. Putting α into the first equation, and multiplying both sides by $(a-c)^2$ gives us

$$(a-c)^2\alpha^2 + a(a-c)\alpha(a-c) + b(a-c)^2 = 0,$$

and hence

$$(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0.$$

For the if-direction, if $a = c$, then $(b-d)^2 = 0$ and hence $b = d$. This means the two quadratic equations are identical, which naturally leads to at least one common root.

2. We first show that the original two equations have at least one common root if and only if $x^2 + ax + b = 0$ and $x^2 + (q-b)x + r = 0$ share a root.

The only-if direction is as follows. Multiplying both sides of the first equation by x, we get

$$x^3 + ax^2 + bx = 0,$$

and hence subtracting this from the second equation gives us

$$x^2 + (q - b)x + r = 0,$$

which means the common root of the original two equations must be a root to the new equation as well.

For the if direction, multiplying both sides of $x^2 + ax + b = 0$ by x and adding this to the new equation gives us that

$$x^3 + (a + 1)x^2 + qx + r = 0.$$

This means the common root of $x^2 + ax + b = 0$ and $x^2 + (q - b)x + r = 0$ must be a root to the cubic equation as well.

Now, the equations $x^2 + ax + b = 0$ and $x^2 + (q - b)x + r = 0$ share a root, if and only if

$$(b - r)^2 - a(b - r)(a - (q - b)) + b(a - (q - b))^2 = 0,$$

which is equivalent to

$$(b - r)^2 - a(b - r)(a + b - q) + b(a + b - q)^2 = 0.$$

The two equations are equivalent to

$$x^2 + \frac{5}{2}x + b = 0, x^3 + \frac{7}{2}x + \frac{5}{2}x + \frac{1}{2} = 0.$$

Let $a = \frac{5}{2}, b = b, q = \frac{5}{2}, r = \frac{1}{2}$, and the two equations have at least common root if and only if

$$\left(b - \frac{1}{2}\right)^2 - \frac{5}{2}\left(b - \frac{1}{2}\right)\left(\frac{5}{2} + b - \frac{5}{2}\right) + b\left(\frac{5}{2} + b - \frac{5}{2}\right)^2 = 0.$$

This simplifies to

$$(2b - 1)^2 - 5(2b - 1)b + 4b^3 = 0,$$

which is equivalent to

$$4b^3 - 6b^2 + b + 1 = 0.$$

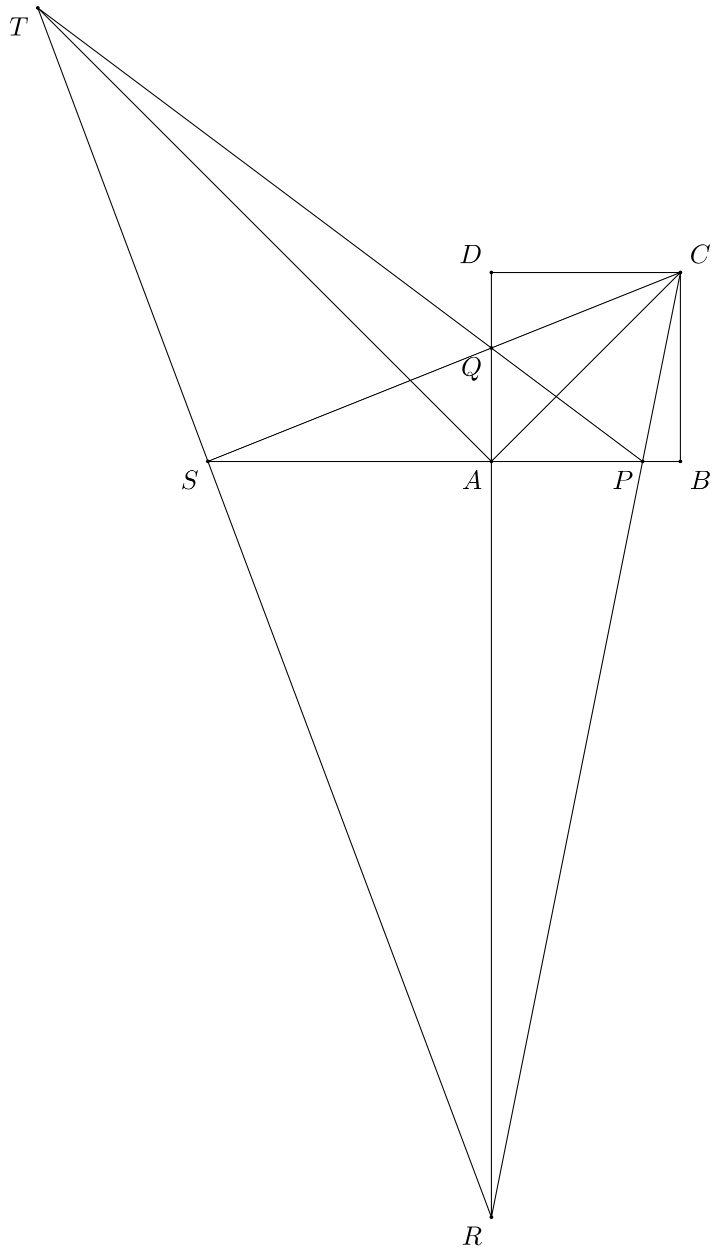
Notice that

$$4b^3 - 6b^2 + b + 1 = (b - 1)(4b^2 - 2b - 1),$$

and hence

$$b_1 = 1, b_{2,3} = \frac{2 \pm \sqrt{2^2 + 4 \cdot 4}}{2 \cdot 4} = \frac{1 \pm \sqrt{5}}{4}.$$

2010.3 Question 5



The line CP has equation

$$l_{CP} : y = \frac{1}{1-n}x - \frac{an}{1-n},$$

and the line DA has equation $l_{DA} : x = 0$. Hence, R has coordinates

$$R\left(0, -\frac{an}{1-n}\right).$$

The line CQ has equation

$$l_{CQ} : y = (1-m)x + am,$$

and the line BA has equation $l_{BA} : y = 0$. Hence, S has coordinates

$$S\left(-\frac{am}{1-m}, 0\right).$$

The line PQ has equation

$$l_{PQ} : y = -\frac{m}{n}x + am,$$

and the line RS has equation

$$l_{RS}: y = -\frac{n(1-m)}{m(1-n)} \cdot x - \frac{an}{1-n}.$$

Therefore, T must have x-coordinates satisfying

$$-\frac{m}{n}x + am = -\frac{n(1-m)}{m(1-n)} \cdot x - \frac{an}{1-n},$$

and hence

$$\left(\frac{n(1-m)}{m(1-n)} - \frac{m}{n}\right) x = -a\left(m + \frac{n}{1-n}\right),$$

and hence

$$\frac{n^2(1-m) - m^2(1-n)}{mn(1-n)} \cdot x = -a\left(\frac{m(1-n)+n}{1-n}\right),$$

which gives

$$\frac{(n+m-mn)(n-m)}{mn(1-n)} \cdot x = -a \cdot \frac{m+n-mn}{1-n}.$$

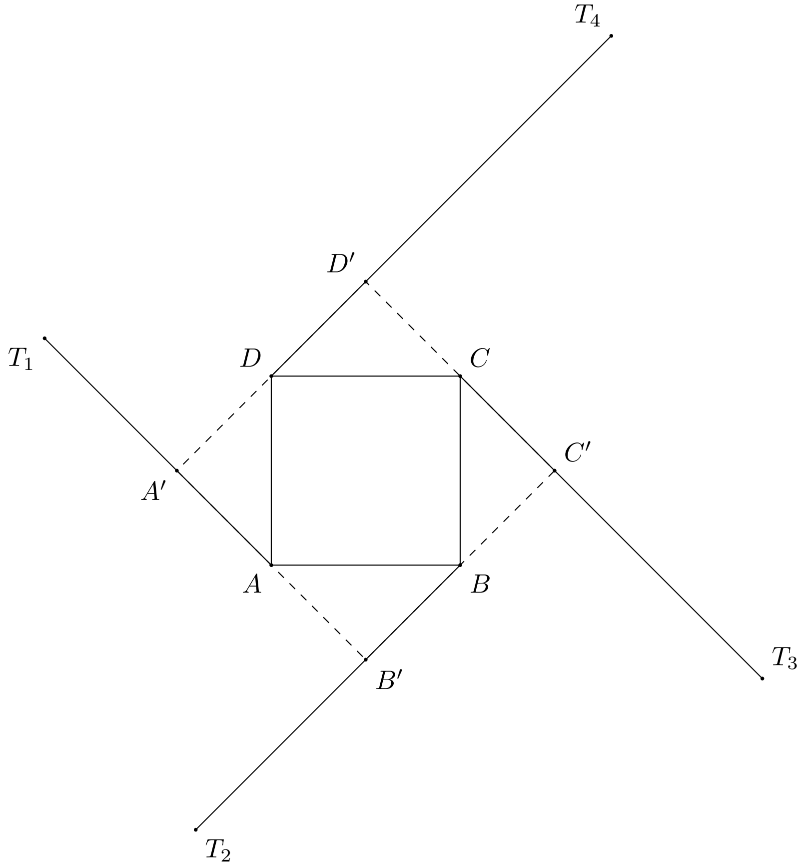
This means

$$x = \frac{amn}{m-n},$$

and hence

$$y = -\frac{m}{n} \cdot \frac{amn}{m-n} + am = \frac{-am^2 + am(m-n)}{m-n} = \frac{-amn}{m-n}.$$

This shows that line AT is the line $y = -x$, while line AC is the line $y = x$. Therefore, means that AT is perpendicular to AC.



Label the square $ABCD$ (in a counter-clockwise sequence), and find two arbitrary points P and Q on AB and AD respectively, with different distances away from A. Construct the line CP and CQ, and let their intersections with AD and AB be R and S respectively. Construct the line RS and PQ, and let them meet at T_1. We have T_1A is perpendicular to AC.

Repeating this process (rotating the labelling of A, B, C and D counter-clockwise), we will get T_2B, T_3C and T_4D, as shown in the diagrams. The square formed by these four lines is $A'B'C'D'$ (found by intersecting the lines). The new square has side length $\sqrt{a}$ equal to the length of the diameter, and hence have area $2a^2$.

2010.3 Question 6

1. The coordinates of P_1 are
$$P_1(\cos\varphi, \sin\varphi, 0),$$
and the coordinates of Q_1 are
$$Q_1(-\sin\varphi, \cos\varphi, 0).$$
Since the rotation is about z-axis, the position of R remains unchanged
$$R_1(0, 0, 1).$$

2. This rotation axis is precisely OQ_1, since it is contained in the x-y plane, and is perpendicular to OP_1. Hence, the position of Q remains unchanged, and hence
$$Q_2(-\sin\varphi, \cos\varphi, 0).$$
If we drop a perpendicular from P_2 to the line OP_1, and call the intersection be P'. We can see from trigonometry that
$$P_2P' = \sin\lambda,$$
and
$$OP' = \cos\lambda.$$
Hence, the x-coordinate of P_2 is $\cos\lambda\cos\varphi$, and the y-coordinate of P_2 is $\cos\lambda\sin\varphi$. The z-coordinate of P_2 is $\sin\lambda$, and hence
$$P_2(\cos\varphi\cos\lambda, \sin\varphi\cos\lambda, \sin\lambda).$$
The relative positions of P, Q and R remains unchanged under rotation, and hence
$$\begin{aligned}
\mathbf{r}_{R_2} &= \mathbf{r}_{P_2} \times \mathbf{r}_{Q_2} \\
&= \begin{pmatrix} \cos\varphi\cos\lambda \\ \sin\varphi\cos\lambda \\ \sin\lambda \end{pmatrix} \times \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos\varphi\cos\lambda & \sin\varphi\cos\lambda & \sin\lambda \\ -\sin\varphi & \cos\varphi & 0 \end{vmatrix} \\
&= \begin{pmatrix} \sin\varphi\cos\lambda\cdot 0 - \sin\lambda\cos\varphi \\ -(\cos\varphi\cos\lambda\cdot 0 + \sin\lambda\cdot\sin\varphi) \\ \cos\varphi\cos\lambda\cdot\cos\varphi + \sin\varphi\cos\lambda\cdot\sin\varphi \end{pmatrix} \\
&= \begin{pmatrix} -\sin\lambda\cos\varphi \\ -\sin\lambda\sin\varphi \\ \cos^2\varphi\cos\lambda + \sin^2\varphi\cos\lambda \end{pmatrix} \\
&= \begin{pmatrix} -\sin\lambda\cos\varphi \\ -\sin\lambda\sin\varphi \\ \cos\lambda \end{pmatrix},
\end{aligned}$$
and hence
$$R_2(-\sin\lambda\cos\varphi, -\sin\lambda\sin\varphi, \cos\lambda).$$

3. The angle of rotation is the angle between OP_0 and OP_2, and hence
$$\begin{aligned}
\cos\theta &= \frac{\overrightarrow{OP_0}\cdot\overrightarrow{OP_2}}{\left|\overrightarrow{OP_0}\right|\cdot\left|\overrightarrow{OP_2}\right|} \\
&= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi\cos\lambda \\ \sin\varphi\cos\lambda \\ \sin\lambda \end{pmatrix} \\
&= \cos\varphi\cos\lambda,
\end{aligned}$$

as desired.

The axis of this rotation must be perpendicular to both OP_1 and OP_2, and hence their cross product

$$\begin{aligned}
\overrightarrow{OP_0} \times \overrightarrow{OP_2} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} \cos\varphi\cos\lambda \\ \sin\varphi\cos\lambda \\ \sin\lambda \end{pmatrix} \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 0 \\ \cos\varphi\cos\lambda & \sin\varphi\cos\lambda & \sin\lambda \end{vmatrix} \\
&= \begin{pmatrix} 0 \\ -\sin\lambda \\ \sin\varphi\cos\lambda \end{pmatrix}
\end{aligned}$$

is a vector in the direction of the axis.

2010.3 Question 7

Since $y = \cos(m \arcsin x)$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{m \sin(m \arcsin x)}{\sqrt{1-x^2}},$$

and

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\frac{m^2 \cos(m \arcsin x) \cdot \frac{1}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2} - m \sin(m \arcsin x) \cdot (-x) \cdot \frac{1}{\sqrt{1-x^2}}}{1-x^2} \\ &= -\frac{m}{1-x^2} \left(m \cos(m \arcsin x) + x \sin(m \arcsin x) \cdot \frac{1}{\sqrt{1-x^2}} \right).\end{aligned}$$

Hence, the left-hand side of the differential equation reduces to

$$\begin{aligned}&(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} - x\frac{\mathrm{d}y}{\mathrm{d}x} + m^2 y \\ &= -m \cdot \left(m \cos(m \arcsin x) + x \sin(m \arcsin x) \cdot \frac{1}{\sqrt{1-x^2}} \right) \\ &\quad + \frac{mx \sin(m \arcsin x)}{\sqrt{1-x^2}} + m^2 \cos(m \arcsin x) \\ &= -m^2 \cos(m \arcsin x) + m^2 \cos(m \arcsin x) \\ &\quad - \frac{mx \sin(m \arcsin x)}{\sqrt{1-x^2}} + \frac{mx \sin(m \arcsin x)}{\sqrt{1-x^2}} \\ &= 0,\end{aligned}$$

as desired.

Differentiating both sides of this equation with respect to x, we get

$$(-2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (1-x^2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - \frac{\mathrm{d}y}{\mathrm{d}x} - x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + m^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

which reduces to

$$(1-x^2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 3x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (m^2-1)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Differentiating both sides with respect to x again, we get

$$(-2x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + (1-x^2)\frac{\mathrm{d}^4y}{\mathrm{d}x^4} - 3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3x\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + (m^2-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0,$$

which reduces to

$$(1-x^2)\frac{\mathrm{d}^4y}{\mathrm{d}x^4} - 5x\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + (m^2-4)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0.$$

The conjecture is for all $n \geq 0$,

$$(1-x^2)\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}} - (2n+1)\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} + (m^2-n^2)\frac{\mathrm{d}^ny}{\mathrm{d}x^n} = 0.$$

The base case where $n = 0$ is already shown. We show the inductive step. Assume this statement is true for some $n = k$, i.e.

$$(1-x^2)\frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} - (2k+1)x\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} + (m^2-k^2)\frac{\mathrm{d}^ky}{\mathrm{d}x^k} = 0.$$

Differentiating both sides with respect to x gives

$$(-2x)\frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} + (1-x^2)\frac{\mathrm{d}^{k+3}y}{\mathrm{d}x^{k+3}} - (2k+1)\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} - (2k+1)x\frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} + (m^2-k^2)\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} = 0,$$

which reduces to

$$(1-x^2)\frac{\mathrm{d}^{k+3}y}{\mathrm{d}x^{k+3}} - (2k+3)x\frac{\mathrm{d}^{k+2}y}{\mathrm{d}x^{k+2}} + (m^2-(k+1)^2)\frac{\mathrm{d}^{k+1}y}{\mathrm{d}x^{k+1}} = 0.$$

This is precisely the statement for when $n = k + 1$.

Hence, by the principle of mathematical induction, the conjecture holds for all integers $n \geq 0$.

Now, we evaluate this at $x = 0$, and we have

$$\left.\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}\right|_{x=0} + (m^2 - n^2) \left.\frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right|_{x=0} = 0$$

for all $n \geq 0$, which rearranged gives

$$\left.\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}\right|_{x=0} = (n^2 - m^2) \left.\frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right|_{x=0}.$$

Notice that

$$y|_{x=0} = \cos(m \arcsin 0) = 1,$$

and

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = -\frac{m \sin(m \arcsin 0)}{\sqrt{1 - 0^2}} = 0$$

Hence,

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} = (0^2 - m^2)\ y|_{x=0} = -m^2,$$

and

$$\left.\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right|_{x=0} = (1^2 - m^2) \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = 0,$$

and

$$\left.\frac{\mathrm{d}^4y}{\mathrm{d}x^4}\right|_{x=0} = (2^2 - m^2) \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} = -m^2(2^2 - m^2),$$

In general, we have

$$\left.\frac{\mathrm{d}^{2n+1}y}{\mathrm{d}x^{2n+1}}\right|_{x=0} = 0,$$

and

$$\left.\frac{\mathrm{d}^{2n}y}{\mathrm{d}x^{2n}}\right|_{x=0} = \prod_{k=0}^{n-1}(4k^2 - m^2) = (-1)^n \prod_{k=0}^{n-1}(m^2 - 4k^2)$$

for all integers $n \geq 0$.

Hence, the Maclaurin series for y satisfy that

$$\begin{aligned} y &= \sum_{n=0}^{\infty} \frac{\left.\frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right|_{x=0} x^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{k=0}^{n-1}(m^2 - 4k^2)x^{2n}}{(2n)!} \\ &= 1 - \frac{m^2x^2}{2!} + \frac{m^2(m^2 - 2^2)x^4}{4!} - \cdots. \end{aligned}$$

In the case where m is even, notice that when $m = 2k$, $m^2 - 4k^2 = 0$, and so for all $n \geq \frac{m}{2} + 1$,

$$\prod_{k=0}^{n-1}(m^2 - 4k^2)x^{2n} = 0,$$

and hence this infinite sum becomes finite:

$$\begin{aligned} y &= \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{k=0}^{n-1}(m^2 - k^2)x^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\frac{m}{2}} \frac{(-1)^n \prod_{k=0}^{n-1}(m^2 - k^2)x^{2n}}{(2n)!}. \end{aligned}$$

Now, let $x = \sin\theta$, we have $\theta = \arcsin x$ since $|\theta| < \frac{1}{2}\pi$, and $y = \cos m\theta$. Hence,

$$\cos m\theta = 1 - \frac{m^2 \sin^2\theta}{2!} + \frac{m^2(m^2 - 2^2)\sin^4\theta}{4!} - \cdots,$$

where the sum is finite (and hence a polynomial), and the degree of this polynomial is m;

2010.3 Question 8

Since $P(x) = Q(x)R'(x) - Q'(x)R(x)$, we notice that

$$\frac{P(x)}{Q(x)^2} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{R(x)}{Q(x)}.$$

Hence,

$$\int \frac{P(x)}{Q(x)^2}\,\mathrm{d}x = \frac{R(x)}{Q(x)} + C$$

where C is a real constant.

1. Since $R(x) = a + bx + cx^2$, we have $R'(x) = b + 2cx$. We let $P(x) = 5x^2 - 4x - 3$ and $Q(x) = 1 + 2x + 3x^2$, and hence $Q'(x) = 6x + 2$.

   Hence,

   $$5x^2 - 4x - 3 = (1 + 2x + 3x^2)(b + 2cx) - (6x + 2)(a + bx + cx^2).$$

   Notice that

   $$\begin{aligned}\mathrm{RHS} &= [6cx^3 + (3b + 4c)x^2 + (2b + 2c)x + b] - [6cx^3 + (6b + 2c)x^2 + (6a + 2b)x + 2a] \\ &= (-3b + 2c)x^2 + (-6a + 2c) + (-2a + b).\end{aligned}$$

   Hence, we have

   $$\begin{cases} & -3b + 2c = 5, \\ -6a + 2c = -4 \iff & 3a - c = 2, \\ & -2a + b = -3. \end{cases}$$

   Notice that

   $$1 \cdot (-3b + 2c) + 2 \cdot (3a - c) + 3 \cdot (-2a + b) = 0,$$

   and

   $$1 \cdot 5 + 2 \cdot 2 - 3 \cdot 3 = 0,$$

   which means that these three equations are linearly dependent. Hence, let $a = 0$, and hence $b = -3$, $c = -2$, $R(x) = -3x - 2x^2$, which gives

   $$\int \frac{5x^2 - 4x - 3}{(1 + 2x + 3x^2)^2}\,\mathrm{d}x = \frac{-3x - 2x^2}{1 + 2x + 3x^2} + C_1.$$

   Letting $a = 1$, and hence $b = -1$, $c = 1$, $R(x) = 1 - x + x^2$, which gives

   $$\int \frac{5x^2 - 4x - 3}{(1 + 2x + 3x^2)^2}\,\mathrm{d}x = \frac{1 - x + x^2}{1 + 2x + 3x^2} + C_2.$$

   Notice that

   $$\frac{1 - x + x^2}{1 + 2x + 3x^2} - \frac{-3x - 2x^2}{1 + 2x + 3x^2} = \frac{1 + 2x + 3x^2}{1 + 2x + 3x^2} = 1,$$

   and the integrals just differ by a constant. Different choices of (a, b, c) lead to results which only differ by a constant.

2. The differential equation we are attempting to solve is equivalent to

   $$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\sin x - 2\cos x}{1 + \cos x + 2\sin x}y = \frac{5 - 3\cos x + 4\sin x}{1 + \cos x + 2\sin x}.$$

   The integrating factor is

   $$\begin{aligned}I(x) &= \exp \int \frac{\sin x - 2\cos x}{1 + \cos x + 2\sin x}\,\mathrm{d}x \\ &= \exp \int -\frac{\mathrm{d}(1 + \cos x + 2\sin x)}{1 + \cos x + 2\sin x} \\ &= \exp(-\ln|1 + \cos x + 2\sin x|) \\ &= \frac{1}{1 + \cos x + 2\sin x},\end{aligned}$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{y}{1+\cos x+2\sin x}=\frac{5-3\cos x+4\sin x}{(1+\cos x+2\sin x)^2}.$$

Let $Q(x)=1+\cos x+2\sin x$, and let $P(x)=5-3\cos x+4\sin x$. We have $Q'(x)=2\cos x-\sin x$ Set $R(x)=a+b\sin x+c\cos x$ for some real constant a,b and c. We have $R'(x)=b\cos x-c\sin x$. Hence,

$$5-3\cos x+4\sin x=(1+\cos x+2\sin x)(b\cos x-c\sin x)-(2\cos x-\sin x)(a+b\sin x+c\cos x).$$

We expand the brackets on the right-hand side, and we have

$$\begin{aligned}\text{RHS}&=b\cos x-c\sin x+b\cos^2 x-c\cos x\sin x+2b\sin x\cos x-2c\sin^2 x\\&\quad-2a\cos x-2b\sin x\cos x-2c\cos^2 x+a\sin x+b\sin^2 x+c\sin x\cos x\\&=(a-c)\sin x+(b-2a)\cos x+(b-2c)\left(\sin^2 x+\cos^2 x\right)\\&=(b-2c)+(a-c)\sin x+(b-2a)\cos x,\end{aligned}$$

and hence by comparing coefficients, we have

$$\begin{cases}b-2c=5,\\a-c=4,\\-2a+b=-3.\end{cases}$$

Notice that

$$1\cdot(b-2c)+(-2)\cdot(a-c)+(-1)\cdot(-2a+b)=0,$$

and

$$1\cdot 5+(-2)\cdot 4+(-1)\cdot(-3)=0,$$

so the system of linear equations is linearly dependent. Hence, set $a=0$, and we have $b=-3, c=-4$, and we have $R(x)=-3\sin x-4\cos x$.

Hence,

$$\begin{aligned}\int\frac{5-3\cos x+4\sin x}{(1+\cos x+2\sin x)^2}&=\int\frac{P(x)}{Q(x)^2}\,\mathrm{d}x\\&=\frac{R(x)}{Q(x)}+C\\&=-\frac{3\sin x+4\cos x}{1+\cos x+2\sin x}+C,\end{aligned}$$

and hence

$$\frac{y}{1+\cos x+2\sin x}=-\frac{3\sin x+4\cos x}{1+\cos x+2\sin x}+C,$$

which means the general solution to the differential equation is

$$y=-(3\sin x+4\cos x)+C(1+\cos x+2\sin x).$$

2010.3 Question 12

Since

$$S = \sum_{n=0}^{\infty} (1 + nd) r^n,$$

we have

$$\begin{aligned}(1-r)S &= S - rS \\ &= \sum_{n=0}^{\infty}(1+nd)r^n - \sum_{n=0}^{\infty}(1+nd)r^{n+1} \\ &= \sum_{n=0}^{\infty}(1+nd)r^n - \sum_{n=1}^{\infty}(1+(n-1)d)r^n \\ &= 1 + \sum_{n=1}^{\infty} dr^n \\ &= 1 + \frac{dr}{1-r},\end{aligned}$$

and hence

$$S = \frac{1}{1-r} + \frac{dr}{(1-r)^2},$$

as desired.

Let X be the number shots taken for Arthur to hit the target for the first time, and we have $X \sim \mathrm{Geo}(a)$, we would like to show $\mathrm{E}(X) = \frac{1}{a}$.

The probability mass function for X satisfies

$$\mathrm{P}(X = x) = (1-a)^{x-1} \cdot a,$$

and hence

$$\begin{aligned}\mathrm{E}(X) &= \sum_{x=1}^{\infty} x\,\mathrm{P}(X = x) \\ &= a \cdot \sum_{x=1}^{\infty} x(1-a)^{x-1} \\ &= a \cdot \sum_{x=0}^{\infty}(1+x)(1-a)^x \\ &= a \cdot \left[\frac{1}{1-(1-a)} + \frac{1\cdot(1-a)}{(1-(1-a))^2}\right] \\ &= a \cdot \left[\frac{1}{a} + \frac{1-a}{a^2}\right] \\ &= a \cdot \frac{1}{a^2} \\ &= \frac{1}{a},\end{aligned}$$

as desired.

Since there is a probability a of Arthur winning on a particular shot, and if Arthur did not hit (with probability $(1-a)$), then there is a probability b of Boadicea winning on the shot, and $(1-b)$ probability that the first two shots both miss, and the game continues as if nothing happened in the first two shots. Therefore,

$$(\alpha, \beta) = a(1,0) + (1-a)b(0,1) + (1-a)(1-b)(\alpha, \beta),$$

and hence

$$\begin{cases}\alpha = a + (1-a)(1-b)\alpha = a + a'b'\alpha, \\ \beta = (1-a)b + (1-a)(1-b)\beta = a' + a'b'\beta,\end{cases} \implies \begin{cases}\alpha = \dfrac{a}{1-a'b'}, \\ \beta = \dfrac{a'b}{1-a'b'}.\end{cases}$$

Let the expected number of shots in the contest be e. By the linearity of the expectation, we have

$$e = a \cdot 1 + a'b \cdot 2 + a'b' \cdot (e + 2),$$

where the $(e + 2)$ comes from when Arthur and Boadicea both miss their initial shots (for the 2), and the game continues (for the e), and hence

$$e = \frac{a + 2a'b + 2a'b'}{1 - a'b'} = \frac{a + 2a'}{1 - a'b'} = \frac{2 - a}{1 - a'b'}.$$

On the other hand, we have

$$\frac{\alpha}{a} + \frac{\beta}{b} = \frac{1}{1 - a'b'} + \frac{1 - a}{1 - a'b'} = \frac{2 - a}{1 - a'b'},$$

and therefore

$$e = \frac{\alpha}{a} + \frac{\beta}{b},$$

as desired.

2010.3 Question 13

Since Z_1 and Z_2 are independent, we have

$$\operatorname{Cov}(Z_1, Z_2) = 0,$$

and hence

$$\operatorname{Corr}(Z_1, Z_2) = \frac{\operatorname{Cov}(Z_1, Z_2)}{\sqrt{\operatorname{Var}(Z_1)\operatorname{Var}(Z_2)}} = \frac{0}{\sqrt{1 \cdot 1}} = 0.$$

For Y_2, we have

$$\begin{aligned}
\mathrm{E}(Y_2) &= \mathrm{E}\left(\rho_{12} Z_1 + \sqrt{1-\rho_{12}^2} Z_2\right) \\
&= \rho_{12}\,\mathrm{E}(Z_1) + \sqrt{1-\rho_{12}^2}\,\mathrm{E}(Z_2) \\
&= \rho_{12} \cdot 0 + \sqrt{1-\rho_{12}^2} \cdot 0 \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\operatorname{Var}(Y_2) &= \operatorname{Var}(\rho_{12} Z_1 + \sqrt{1-\rho_{12}^2} Z_2) \\
&= \rho_{12}^2 \operatorname{Var}(Z_1) + (1-\rho_{12}^2)\operatorname{Var}(Z_2) \\
&= \rho_{12}^2 \cdot 1 + (1-\rho_{12}^2) \cdot 1 \\
&= 1,
\end{aligned}$$

and hence

$$\begin{aligned}
\operatorname{Corr}(Y_1, Y_2) &= \frac{\operatorname{Cov}(Y_1, Y_2)}{\sqrt{\operatorname{Var}(Y_1)\operatorname{Var}(Y_2)}} \\
&= \frac{\mathrm{E}(Y_1 Y_2) - \mathrm{E}(Y_1)\,\mathrm{E}(Y_2)}{\sqrt{1 \cdot 1}} \\
&= \mathrm{E}\left(\rho_{12} Z_1^2 + \rho_{12}\sqrt{1-\rho_{12}^2} Z_1 Z_2\right) - 0 \cdot 0 \\
&= \rho_{12}\,\mathrm{E}(Z_1^2) + \rho_{12}\sqrt{1-\rho_{12}^2}\,\mathrm{E}(Z_1 Z_2) \\
&= \rho_{12}\left(\operatorname{Var}(Z_1) + \mathrm{E}(Z_1)^2\right) + \rho_{12}\sqrt{1-\rho_{12}^2}\,\mathrm{E}(Z_1)\,\mathrm{E}(Z_2) \\
&= \rho_{12}\left(1 + 0^2\right) + \rho_{12}\sqrt{1-\rho_{12}^2} \cdot 0 \cdot 0 \\
&= \rho_{12}.
\end{aligned}$$

For Y_3, we have

$$\begin{aligned}
\operatorname{Var}(Y_3) &= \operatorname{Var}(aZ_1 + bZ_2 + cZ_3) \\
&= a^2 \operatorname{Var}(Z_1) + b^2 \operatorname{Var}(Z_2) + c^2 \operatorname{Var}(Z_3) \\
&= a^2 + b^2 + c^2 \\
&= 1,
\end{aligned}$$

and hence $a^2 + b^2 + c^2 = 1$.

For the correlation, we have

$$\begin{aligned}
\operatorname{Corr}(Y_1, Y_3) &= \frac{\operatorname{Cov}(Y_1 Y_3)}{\sqrt{\operatorname{Var}(Y_1)\operatorname{Var}(Y_3)}} \\
&= \frac{\operatorname{E}(Y_1 Y_3) - \operatorname{E}(Y_1)\operatorname{E}(Y_3)}{\sqrt{1 \cdot 1}} \\
&= \frac{\operatorname{E}(aZ_1^2 + bZ_1 Z_2 + cZ_1 Z_3) - 0 \cdot 0}{1} \\
&= a\operatorname{E}(Z_1^2) + b\operatorname{E}(Z_1 Z_2) + c\operatorname{E}(Z_1 Z_3) \\
&= a(\operatorname{Var}(Z_1) + \operatorname{E}(Z_1)^2) + b\operatorname{E}(Z_1)\operatorname{E}(Z_2) + c\operatorname{E}(Z_1)\operatorname{E}(Z_3) \\
&= a(1 + 0^2) + b \cdot 0 \cdot 0 + c \cdot 0 \cdot 0 \\
&= a \\
&= \rho_{13},
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Corr}(Y_2, Y_3) &= \frac{\operatorname{Cov}(Y_2 Y_3)}{\sqrt{\operatorname{Var}(Y_2)\operatorname{Var}(Y_3)}} \\
&= \frac{\operatorname{E}(Y_2 Y_3) - \operatorname{E}(Y_2)\operatorname{E}(Y_3)}{\sqrt{1 \cdot 1}} \\
&= \frac{\operatorname{E}\left((aZ_1 + bZ_2 + cZ_3) \cdot \left(\rho_{12} Z_1 + \sqrt{1 - \rho_{12}^2 Z_2}\right)\right) - 0 \cdot 0}{1} \\
&= \operatorname{E}\left(a\rho_{12} Z_1^2 + b\sqrt{1 - \rho_{12}^2} Z_2^2\right) \\
&= a\rho_{12}(\operatorname{Var}(Z_1) + \operatorname{E}(Z_1)^2) + b\sqrt{1 - \rho_{12}^2}(\operatorname{Var}(Z_2) + \operatorname{E}(Z_2)^2) \\
&= a\rho_{12} + b\sqrt{1 - \rho_{12}^2} \\
&= \rho_{23},
\end{aligned}$$

since all the cross-term expectation is 0, i.e. for $i \neq j$, $\operatorname{E}(Z_i Z_j) = \operatorname{E}(Z_i)\operatorname{E}(Z_j) = 0$. Hence,

$$b = \frac{\rho_{23} - \rho_{12}\rho_{13}}{\sqrt{1 - \rho_{12}^2}},$$

and therefore,

$$c = \sqrt{1 - a^2 - b^2} = \sqrt{1 - \rho_{13}^2 - \frac{(\rho_{23} - \rho_{12}\rho_{13})^2}{1 - \rho_{12}^2}}.$$

We could have $X_i = \mu_i + \sigma_i Y_i$ for $i = 1, 2, 3$, since

$$\operatorname{E}(X_i) = \mu_i + \sigma_i \operatorname{E}(Y_i) = \mu_i + \sigma_i \cdot 0 = \mu_i,$$

and

$$\operatorname{Var}(X_i) = \sigma_i^2 \operatorname{Var}(Y_i) = \sigma_i^2 \cdot 1 = \sigma_i^2.$$

As for correlation, we notice that for any random variables U, V, we have

$$\begin{aligned}
\mathrm{Corr}(aU+b, cU+d) &= \frac{\mathrm{Cov}(aU+b, cV+d)}{\sqrt{\mathrm{Var}(aU+b)\,\mathrm{Var}(cV+d)}} \\
&= \frac{\mathrm{E}((aU+b)(cV+d)) - \mathrm{E}(aU+b)\,\mathrm{E}(cV+d)}{\sqrt{a^2\,\mathrm{Var}(U)c^2\,\mathrm{Var}(V)}} \\
&= \frac{ac\,\mathrm{E}(UV) + bc\,\mathrm{E}(V) + ad\,\mathrm{E}(U) + bd - (a\,\mathrm{E}(U)+b)(c\,\mathrm{E}(V)+d)}{ac\sqrt{\mathrm{Var}(U)\,\mathrm{Var}(V)}} \\
&= \frac{ac\,\mathrm{E}(UV) + bc\,\mathrm{E}(V) + ad\,\mathrm{E}(U) + bd - ac\,\mathrm{E}(U)\,\mathrm{E}(V) - bc\,\mathrm{E}(V) - ad\,\mathrm{E}(U) - bd}{ac\sqrt{\mathrm{Var}(U)\,\mathrm{Var}(V)}} \\
&= \frac{ac(\mathrm{E}(UV) - \mathrm{E}(U)\,\mathrm{E}(V))}{ac\sqrt{\mathrm{Var}(U)\,\mathrm{Var}(V)}} \\
&= \frac{\mathrm{Cov}(U, V)}{\sqrt{\mathrm{Var}(U)\,\mathrm{Var}(V)}} \\
&= \mathrm{Corr}(U, V),
\end{aligned}$$

which shows linear coding does not affect the correlation. This implies

$$\mathrm{Corr}(X_i, X_j) = \mathrm{Corr}(Y_i, Y_j) = \rho_{ij}$$

for $i \neq j$. Therefore, $X_i = \mu_i + \sigma_i Y_i$ for $i = 1, 2, 3$ satisfies the desired.