2024.3 Question 7

1. For the left inequality, $f(n) > 0$ since $f(n) > \frac{1}{n+1} > 0$.

   For the right inequality, we notice that

$$\begin{aligned}
f(n) &= \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots \\
&< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots \\
&= \frac{1}{n+1} \cdot \frac{1}{1 - \frac{1}{n+1}} \\
&= \frac{1}{(n+1) - 1} \\
&= \frac{1}{n}.
\end{aligned}$$

   Hence,

$$0 < f(n) < \frac{1}{n}.$$

2. For the left inequality, by grouping consecutive terms, we have

$$\begin{aligned}
g(n) &= \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} \\
&\quad + \frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)} + \cdots \\
&= \left(\frac{1}{n+1} - \frac{1}{(n+1)(n+2)}\right) \\
&\quad + \left(\frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)}\right) + \cdots \\
&> \left(\frac{1}{n-1} - \frac{1}{n+1}\right) \\
&\quad + \left(\frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)}\right) + \cdots \\
&= 0 + 0 + \cdots \\
&= 0,
\end{aligned}$$

   using the inequality

$$\frac{1}{(n+1)\cdots(n+k)} > \frac{1}{(n+1)\cdots(n+k)(n+k+1)}.$$

For the right inequality, by grouping consecutive after the first one, we have

$$\begin{aligned}
g(n) &= \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} \\
&\quad - \frac{1}{(n+1)(n+2)(n+3)(n+4)} + \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)} - \cdots \\
&= \frac{1}{n+1} - \left(\frac{1}{(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)}\right) \\
&\quad - \left(\frac{1}{(n+1)(n+2)(n+3)(n+4)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)}\right) - \cdots \\
&< \frac{1}{n+1} - \left(\frac{1}{(n+1)(n+2)} - \frac{1}{(n+1)(n+2)}\right) \\
&\quad - \left(\frac{1}{(n+1)(n+2)(n+3)(n+4)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)}\right) - \cdots \\
&= \frac{1}{n+1} - 0 - 0 - \cdots \\
&= \frac{1}{n+1},
\end{aligned}$$

using the inequality

$$\frac{1}{(n+1)\cdots(n+k-1)(n+k)} < \frac{1}{(n+1)\cdots(n+k-1)}.$$

Hence,

$$0 < g(n) < \frac{1}{n+1}.$$

3. The infinite series for e is given by

$$e = \sum_{t=0}^{\infty} \frac{1}{t!},$$

and notice that

$$f(n) = \sum_{t=1}^{\infty} \frac{n!}{(n+t)!} = n! \sum_{t=1}^{\infty} \frac{1}{(n+t)!}.$$

Hence,

$$\begin{aligned}
(2n)!e - f(2n) &= (2n)! \sum_{t=0}^{\infty} \frac{1}{t!} - (2n)! \sum_{t=1}^{\infty} \frac{1}{(2n+t)!} \\
&= (2n)! \left(\sum_{t=0}^{\infty} \frac{1}{t!} - \sum_{t=2n+1}^{\infty} \frac{1}{t!}\right) \\
&= (2n)! \sum_{t=0}^{2n} \frac{1}{t!} \\
&= \sum_{t=0}^{2n} \frac{(2n)!}{t!}.
\end{aligned}$$

Since $t \leq 2n$, the terms in the sum represents the number of ways to arrange $(2n-t)$ items out of $2n$ items, which must be integers. Hence, the sum is an integer as well.

Similarly, the infinite series for e^{-1} is given by

$$e^{-1} = \sum_{t=0}^{\infty} \frac{(-1)^t}{t!},$$

and notice that

$$g(n) = -\sum_{t=1}^{\infty} \frac{(-1)^t n!}{(n+t)!} = -n! \sum_{t=1}^{\infty} \frac{(-1)^t}{(n+t)!}.$$

Hence,

$$\begin{aligned}
\frac{(2n)!}{e} + g(2n) &= (2n)! \sum_{t=0}^{\infty} \frac{(-1)^t}{t!} - (2n)! \sum_{t=1}^{\infty} \frac{(-1)^t}{(n+t)!} \\
&= (2n)! \left(\sum_{t=0}^{\infty} \frac{(-1)^t}{t!} - \sum_{t=2n+1}^{\infty} \frac{(-1)^t}{t!} \right) \\
&= (2n)! \sum_{t=0}^{2n} \frac{(-1)^t}{t!} \\
&= \sum_{t=0}^{2n} \frac{(-1)^t (2n)!}{t!},
\end{aligned}$$

and by the same argument, since $t \leq 2n$, this must be an integer as well.

4. By the previous part, let $a(n) = f(2n) - (2n)!e$, and $b(n) = g(2n) + \frac{(2n)!}{e}$, we must have that $a, b : \mathbb{N} \to \mathbb{Z}$ since they are integers.

   Using this notation,

$$\begin{aligned}
qf(2n) + pg(2n) &= qa(2n) + qe(2n)! + pb(2n) - \frac{p}{e}(2n)! \\
&= qa(2n) + pb(2n) + \left(qe - \frac{p}{e}\right)(2n)! \\
&= qa(2n) + pb(2n)
\end{aligned}$$

   must be an integer, since $p, q, a(2n), b(2n)$ are all integers.

5. Assume B.W.O.C. that e^2 is irrational. Then there exists natural numbers p, q such that

$$e^2 = \frac{p}{q} \iff qe = \frac{p}{e}.$$

   Since $e^2 > 1$, $p > q$.

   On one hand, we have $qf(2n) + pg(2n) > 0$.

   On the other hand, let $n = p$,

$$\begin{aligned}
qf(2n) + pg(2n) &< q \cdot \frac{1}{2p} + p \cdot \frac{1}{2p+1} \\
&< q \cdot \frac{1}{2p} + p \cdot \frac{1}{2p} \\
&= \frac{p+q}{2p} \\
&< \frac{2p}{2p} \\
&= 1.
\end{aligned}$$

   This means

$$0 < qf(2p) + pg(2p) < 1.$$

   But by the previous part, $qf(2n) + pg(2n)$ is an integer for all positive integer n, and $n = p$ is a positive integer. This leads to a contradiction.

   Hence, such p and q does not exist, meaning e^2 is not rational, hence e^2 is irrational.