2024.3 Question 6

1. (a) We have

$$\begin{aligned}\frac{\mathrm{d}x-y}{\mathrm{d}t} &= \frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}y}{\mathrm{d}t} \\ &= (-x+3y+u)-(x+y+u) \\ &= -2x+2y \\ &= -2(x-y).\end{aligned}$$

This is a differential equation for $x-y$ in terms of t, and hence it solves to

$$x-y=Ae^{-2t}.$$

If $x=y=0$ for some $t>0$, then it must be the case that $A=0$, giving $x-y=0$, and $x=y$. Therefore, for $t=0$, we must also necessarily have $x_0=y_0$.

(b) Given that $x_0=y_0$, we must have $x=y$ for all $t>0$. Hence,

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -x+3x+u \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= 2x+u \\ \frac{\mathrm{d}x}{2x+u} &= \mathrm{d}t \\ \ln|2x+u| &= 2t+C \\ 2x+u &= Ae^{2t}.\end{aligned}$$

Since at $t=0$, $x=x_0$, we must have $A=2x_0+u$, and hence

$$2x+u=(2x_0+u)e^{2t},$$

and rearranging gives

$$u=\frac{2(x_0e^{2t}-x)}{1-e^{2t}}.$$

The particle is at origin at time $t=T>0$, and hence $x=y=0$ for $t=T$, and hence

$$u=\frac{2x_0e^{2T}}{1-e^{2T}}.$$

This ensures the particle is at origin as well since this ensures the particle is at $x=0$ for $t=T$, and $y=x$ so $y=0$ as well.

2. (a) Consider $\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}z}{\mathrm{d}t}-2\frac{\mathrm{d}y}{\mathrm{d}t}$, and we have

$$\begin{aligned}\frac{\mathrm{d}x+z-2y}{\mathrm{d}t} &= \frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\mathrm{d}z}{\mathrm{d}t}-2\frac{\mathrm{d}y}{\mathrm{d}t} \\ &= (4y-5z+u)+(x-2y+u)-2(x-2z+u) \\ &= 4y-5z+u+x-2y+u-2x+4z-2u \\ &= -x-z+2y,\end{aligned}$$

and hence

$$x+z-2y=Ae^{-t}.$$

Since the particle is at the origin at some time $t>0$, we must have $A=0$, and hence

$$x+z-2y=0,$$

which means $y=\frac{x+z}{2}$ for all time t.
At time $t=0$, $y_0=\frac{x_0+z_0}{2}$, and so y_0 is the mean of x_0 and z_0.

(b) Since $2y = x + z$, we must have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2(x + z) - 5z + u = 2x - 3z + u,$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = x - (x + z) + u = -z + u.$$

Hence, considering $\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}z}{\mathrm{d}t}$, we have

$$\begin{aligned}\frac{\mathrm{d}x - z}{\mathrm{d}t} &= \frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}z}{\mathrm{d}t} \\ &= (2x - 3z + u) - (-z + u) \\ &= 2(x - z),\end{aligned}$$

which gives

$$x - z = Ae^{2t}.$$

Since the particle is at the origin for some $t > 0$, we must have $A = 0$. This means $x = z$ for all t, and further we have $x = y = z$ for all t since $2y = x + z$.

At $t = 0$, this means $x_0 = y_0 = z_0$ as desired.

(c) Given that $x_0 = y_0 = z_0$, all previous parts still apply, since the boundary condition of $2y = x + z$ and $x = z$ holds for $t = 0$. Hence, $x = y = z$ for all t, and

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -x + u \\ \frac{\mathrm{d}x}{x - u} &= -\,\mathrm{d}t \\ \ln|x - u| &= -t + C \\ x - u &= Ae^{-t}.\end{aligned}$$

At $t = 0$, $x = x_0$, we must have $A = x_0 - u$, and hence

$$x - u = (x_0 - u)e^{-t},$$

and rearranging gives

$$u = \frac{x_0 e^{-t} - x}{1 - e^{-t}}.$$

The particle is at origin at a time $t = T > 0$, and hence $x = y = z = 0$ for $t = T$, and hence

$$u = \frac{x_0 e^{-T}}{1 - e^{-T}} = \frac{x_0}{1 + e^{T}}.$$

This ensures the particle is at origin as well since this ensures the particle is at $x = 0$ for $t = T$, and $x = y = z$, so $y = z = 0$ as well.