2024.3 Question 5

1. Let

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \mathbf{N} = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

and hence we have

$$\operatorname{tr} \mathbf{M} = a + d, \operatorname{tr} \mathbf{N} = e + h.$$

Notice that

$$\mathbf{MN} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}, \mathbf{NM} = \begin{pmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{pmatrix},$$

which means

$$\operatorname{tr}(\mathbf{MN}) = ae + bg + cf + dh, \operatorname{tr}(\mathbf{NM}) = ae + cf + bg + dh,$$

and hence $\operatorname{tr}(\mathbf{MN}) = \operatorname{tr}(\mathbf{NM})$ as desired.

We also have

$$\mathbf{M} + \mathbf{N} = \begin{pmatrix} a + e & b + f \\ c + g & d + h \end{pmatrix},$$

meaning $\operatorname{tr}(\mathbf{M} + \mathbf{N}) = a + e + d + h = (a + d) + (e + h) = \operatorname{tr} \mathbf{M} + \operatorname{tr} \mathbf{N}$.

2. We have $\det \mathbf{M} = ad - bc$, and hence

$$\frac{\mathrm{d}}{\mathrm{d}t} \det \mathbf{M} = \dot{a}d + a\dot{d} - \dot{b}c - b\dot{c}.$$

Hence,

$$\text{LHS} = \frac{1}{ad - bd} \left(\dot{a}d + a\dot{d} - \dot{b}c - b\dot{c} \right).$$

On the other hand,

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \begin{pmatrix} \dot{a} & \dot{b} \\ \dot{c} & \dot{d} \end{pmatrix}, \mathbf{M}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and hence

$$\begin{aligned} \mathbf{M}^{-1} \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} &= \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \dot{a} & \dot{b} \\ \dot{c} & \dot{d} \end{pmatrix} \\ &= \frac{1}{ad - bc} \begin{pmatrix} \dot{a}d - b\dot{c} & \dot{b}d - b\dot{d} \\ -\dot{a}c + a\dot{c} & -\dot{b}c + a\dot{d} \end{pmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} \text{RHS} &= \operatorname{tr} \left(\mathbf{M}^{-1} \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} \right) \\ &= \frac{1}{ad - bc} \left(\dot{a}d - b\dot{c} - \dot{b}c + a\dot{d} \right) \\ &= \frac{1}{ad - bc} \left(\dot{a}d + a\dot{d} - b\dot{c} - \dot{b}c \right) \\ &= \text{LHS}, \end{aligned}$$

as desired.

3. $\det \mathbf{M} \neq 0$ since $\mathbf{M}$ is non-singular, and hence left-multiplying by $\mathbf{M}^{-1}$ on both sides gives us

$$\mathbf{M}^{-1} \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \mathbf{N} - \mathbf{M}^{-1}\mathbf{N}\mathbf{M}.$$

Taking trace on both sides, we have

$$\begin{aligned}\frac{1}{\det \mathbf{M}} \frac{\mathrm{d}}{\mathrm{d}t} \det \mathbf{M} &= \operatorname{tr}\left(\mathbf{M}^{-1} \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t}\right) \\ &= \operatorname{tr}\left(\mathbf{N} - \mathbf{M}^{-1}\mathbf{N}\mathbf{M}\right) \\ &= \operatorname{tr}\mathbf{N} - \operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{N}\mathbf{M}\right) \\ &= \operatorname{tr}\mathbf{N} - \operatorname{tr}\left(\left(\mathbf{M}^{-1}\mathbf{N}\right)\mathbf{M}\right) \\ &= \operatorname{tr}\mathbf{N} - \operatorname{tr}\left(\mathbf{M}\left(\mathbf{M}^{-1}\mathbf{N}\right)\right) \\ &= \operatorname{tr}\mathbf{N} - \operatorname{tr}\left(\left(\mathbf{M}\mathbf{M}^{-1}\right)\mathbf{N}\right) \\ &= \operatorname{tr}\mathbf{N} - \operatorname{tr}\left(\mathbf{I}\mathbf{N}\right) \\ &= \operatorname{tr}\mathbf{N} - \operatorname{tr}\mathbf{N} \\ &= 0.\end{aligned}$$

Hence, $\frac{\mathrm{d}}{\mathrm{d}t} \det \mathbf{M} = 0$, which means $\det \mathbf{M}$ is a constant independent of t.
Directly taking trace on both sides, we have

$$\begin{aligned}\operatorname{tr}\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} &= \operatorname{tr}(\mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M}) \\ &= \operatorname{tr}(\mathbf{M}\mathbf{N}) - \operatorname{tr}(\mathbf{N}\mathbf{M}) \\ &= 0,\end{aligned}$$

and note

$$\operatorname{tr}\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{tr}\mathbf{M},$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{tr}\mathbf{M} = 0,$$

meaning $\operatorname{tr}\mathbf{M}$ is a constant independent of t.
Notice that

$$\operatorname{tr}\left(\mathbf{M}^2\right) = \operatorname{tr}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = a^2 + bc + bc + d^2 = a^2 + 2bc + d^2.$$

Since $\operatorname{tr}\mathbf{M}$ and $\det\mathbf{M}$ are both independent of t, we must have

$$\begin{aligned}(\operatorname{tr}\mathbf{M})^2 - 2\det\mathbf{M} &= (a+d)^2 - 2(ad - bc) \\ &= a^2 + 2ad + d^2 - 2ad + 2bc \\ &= a^2 + 2bc + d^2 \\ &= \operatorname{tr}\left(\mathbf{M}^2\right)\end{aligned}$$

is independent of t as well.
Let

$$\mathbf{M} = \begin{pmatrix} A + x & b \\ c & D - x \end{pmatrix},$$

the diagonal ones being so since the trace is independent of t. Here, x is a function of t.
By differentiating,

$$\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \begin{pmatrix} \dot{x} & \dot{b} \\ \dot{c} & -\dot{x} \end{pmatrix},$$

and the right-hand side satisfies

$$\begin{aligned}\mathbf{M}\mathbf{N} - \mathbf{N}\mathbf{M} &= \begin{pmatrix} A + x & b \\ c & D - x \end{pmatrix}\begin{pmatrix} t & t \\ & t \end{pmatrix} - \begin{pmatrix} t & t \\ & t \end{pmatrix}\begin{pmatrix} A + x & b \\ c & D - x \end{pmatrix} \\ &= \begin{pmatrix} t(A+x) & (A+x)t + bt \\ ct & ct + (D-x)t \end{pmatrix} - \begin{pmatrix} t(A+x) + ct & bt + t(D-x) \\ ct & t(D-x) \end{pmatrix} \\ &= \begin{pmatrix} -ct & (A - D + 2x)t \\ 0 & ct. \end{pmatrix}\end{aligned}$$

Comparing the components, we see that $\dot{c} = 0$, meaning that c is a constant: $c = C$.
Hence, $\dot{x} = -Ct$, which solves to $x = -\frac{Ct^2}{2}$, since $x = 0$ when $t = 0$.
This means
$$\dot{b} = (A - D + 2x)t = (A - D - Ct^2)t,$$
and hence
$$b = \frac{(A-D)t^2}{2} - \frac{Ct^4}{4} + B$$
since $b = B$ when $t = 0$.
Hence,
$$\mathbf{M} = \begin{pmatrix} A - Ct^2/2 & (A-D)t^2/2 - Ct^4/4 \\ C & D + Ct^2/2 \end{pmatrix}$$
is the solution given the conditions.

4. By rearranging, we have
$$\mathbf{N} = \mathbf{M}^{-1}\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t}.$$
Hence, let
$$\mathbf{M} = \begin{pmatrix} 1 + e^t & \\ & 1 - e^t \end{pmatrix},$$
we have
$$\operatorname{tr}\mathbf{M} = 2$$
which is non-zero and independent of t.
Hence,
$$\mathbf{M}^{-1} = \frac{1}{1 - e^{2t}}\begin{pmatrix} 1 - e^t & \\ & 1 + e^t \end{pmatrix}, \frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} = \begin{pmatrix} e^t & \\ & -e^t \end{pmatrix},$$
so
$$\begin{aligned}\mathbf{N} &= \frac{1}{1 - e^{2t}}\begin{pmatrix} 1 - e^t & \\ & 1 + e^t \end{pmatrix}\begin{pmatrix} e^t & \\ & -e^t \end{pmatrix} \\ &= \frac{1}{1 - e^{2t}}\begin{pmatrix} e^t(1 - e^t) & \\ & -e^t(1 + e^t) \end{pmatrix},\end{aligned}$$
which gives
$$\operatorname{tr}\mathbf{N} = \frac{e^{2t}}{e^{2t} - 1}$$
which is clearly non-zero.