2024.3 Question 4

1. The angle between a line with gradient m and the positive x-axis is $\arctan m$. Hence, we must have

$$\begin{aligned}\arctan m_1 - \arctan m_2 &= \pm\frac{\pi}{4}\\ \tan\left(\arctan m_1 - \arctan m_2\right) &= \tan\left(\pm\frac{\pi}{4}\right)\\ \frac{m_1 - m_2}{1 + m_1 m_2} &= \pm 1,\end{aligned}$$

as desired.

2. We have $y = \frac{x^2}{4a}$, and hence $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{2a}$. Hence, the tangent to the point $\left(p, \frac{p^2}{4a}\right)$ is given by

$$\begin{aligned}y - \frac{p^2}{4a} &= \frac{p}{2a}(x - p)\\ 4ay - p^2 &= 2p(x - p)\\ 4ay &= 2px - p^2,\end{aligned}$$

with gradient $\frac{2p}{4a} = \frac{p}{2a}$, and the tangent to the point $\left(q, \frac{q^2}{4a}\right)$ is given by $4ay = 2qx + q^2$, with gradient $\frac{q}{2a}$.
Hence, when they intersect, it must be the case that

$$\begin{aligned}2px - p^2 &= 2qx - q^2\\ 2(p-q)x &= p^2 - q^2\\ 2(p-q)x &= (p+q)(p-q)\\ x &= \frac{p+q}{2}\end{aligned}$$

since $p \neq q$.
The y-coordinate is given by

$$\begin{aligned}y &= \frac{2px - p^2}{4a}\\ &= \frac{p^2 + pq - p^2}{4a}\\ &= \frac{pq}{4a}.\end{aligned}$$

If the two curves meet at $\frac{\pi}{4}$, the gradients must satisfy that

$$\begin{aligned}\frac{\frac{p}{2a} - \frac{q}{2a}}{1 + \frac{p}{2a}\cdot\frac{q}{2a}} &= \pm 1\\ \frac{2a(p-q)}{4a^2 + pq} &= \pm 1\\ 2a(p-q) &= \pm\left(4a^2 + pq\right)\\ 4a^2(p-q)^2 &= (4a^2 + pq)^2\\ 4a^2p^2 - 8a^2pq + 4a^2q^2 &= 16a^4 + 8a^2pq + p^2q^2\\ p^2q^2 + 16a^2pq + 16a^4 - 4a^2p^2 - 4a^2q^2 &= 0.\end{aligned}$$

For the intersection of the two tangents, we consider $(y + 3a)^2 - (8a^2 + x^2)$.

$$\begin{aligned}(y+3a)^2 - (8a^2 + x^2) &= y^2 + 6ay + 9a^2 - 8a^2 - x^2\\ &= y^2 + 6ay - x^2 + a^2\\ &= \frac{p^2q^2}{16a^2} + 6a\cdot\frac{pq}{4a} - \left(\frac{p+q}{2}\right)^2 + a^2\\ &= \frac{p^2q^2}{16a^2} + \frac{3pq}{2} - \frac{(p+q)^2}{4} + a^2.\end{aligned}$$

We have the following being equivalent:

$$\begin{aligned}
(y+3a)^2 &= 8a^2+x^2 \\
\frac{p^2q^2}{16a^2}+\frac{3pq}{2}-\frac{(p+q)^2}{4}+a^2 &= 0 \\
p^2q^2+3pq\cdot 8a^2-(p+q)^2\cdot 4a^2+a^2\cdot 16a^2 &= 0 \\
p^2q^2+24pqa^2-4a^2p^2-4a^2q^2-8pqa^2+16a^4 &= 0 \\
p^2q^2+16a^2pq+16a^4-4a^2p^2-4a^2q^2 &= 0,
\end{aligned}$$

which was true due to the tangents intersecting at $\frac{\pi}{4}$.

Hence, we must have the intersection of two tangents lie on $(y+3a)^2 = 8a^2+x^2$, which finishes our proof.

3. Let θ be this acute angle, and from the previous part, we can see that

$$\begin{aligned}
4a^2(p-q)^2 &= \tan^2\theta(4a^2+pq)^2 \\
4a^2p^2-8a^2pq+4a^2q^2 &= \tan^2\theta 16a^4+\tan^2\theta 8a^2pq+\tan^2\theta p^2q^2 \\
\tan^2\theta p^2q^2+8(\tan^2\theta+1)a^2pq+\tan^2\theta 16a^4 &= 4a^2p^2+4a^2q^2
\end{aligned}$$

Given $(y+7a)^2 = 48a^2+3x^2$ for the intersection of the two tangents, we have

$$\begin{aligned}
(y+7a)^2-\left(48a^2+3x^2\right) &= 0 \\
\left(\frac{pq}{4a}+7a\right)^2-\left(48a^2+3\left(\frac{p+q}{2}\right)^2\right) &= 0 \\
\frac{p^2q^2}{16a^2}+\frac{7pq}{2}+49a^2-48a^2-\frac{3(p+q)^2}{4} &= 0 \\
p^2q^2+8a^2\cdot 7pq+16a^4-3(p+q)^2\cdot 4a^2 &= 0 \\
p^2q^2+56pqa^2+16a^4-12p^2a^2-12q^2a^2-24pqa^2 &= 0 \\
p^2q^2+32pqa^2+16a^4-12p^2a^2-12q^2a^2 &= 0 \\
p^2q^2+32pqa^2+16a^4-3\left(\tan^2\theta p^2q^2+8(\tan^2\theta+1)a^2pq+16\tan^2\theta a^4\right) &= 0 \\
(1-3\tan^2\theta)p^2q^2+8(1-3\tan^2\theta)pqa^2+16(1-3\tan^2\theta)a^4 &= 0 \\
(1-3\tan^2\theta)\left[p^2q^2+8pqa^2+16a^4\right] &= 0 \\
(1-3\tan^2\theta)(pq+4a^2)^2 &= 0.
\end{aligned}$$

Hence, either $pq+4a^2=0$, or $1-3\tan^2\theta=0$. The former cannot always the case. Therefore, $1-3\tan^2\theta=0$, which gives $\tan\theta=\pm\frac{\sqrt{3}}{3}$.

Since θ is acute, we have $\tan\theta=\frac{\sqrt{3}}{3}$, and hence $\theta=\frac{\pi}{6}$ is the acute angle between the two tangents.