2024.3 Question 3

1. (a) Notice that by partial fractions, we have

$$\frac{x+c}{x(x+1)} = \frac{1-c}{x+1} + \frac{c}{x}.$$

Hence, by differentiating, we have

$$\begin{aligned} g'(x) &= \frac{1}{1+\frac{1}{x}} \cdot \left(-\frac{1}{x^2}\right) + \frac{1-c}{(x+1)^2} + \frac{c}{x^2} \\ &= -\frac{1}{x^2+x} + \frac{1-c}{(x+1)^2} + \frac{c}{x^2} \\ &= \frac{-x(x+1) + (1-c)x^2 + c(x+1)^2}{(x+1)^2x^2} \\ &= \frac{cx^2 + 2cx + c + x^2 - cx^2 - x^2 - x}{(x+1)^2x^2} \\ &= \frac{(2c-1)x + c}{(x+1)^2x^2}. \end{aligned}$$

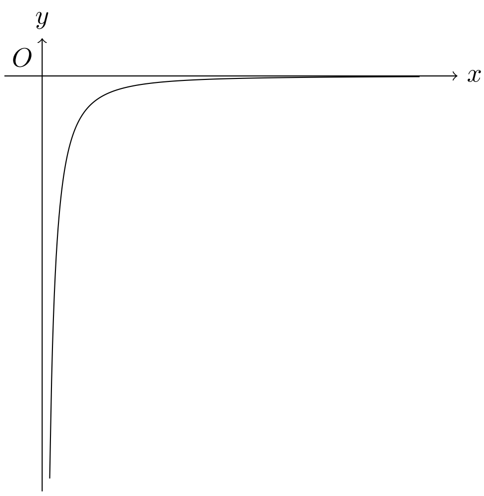
Given that $c \geq \frac{1}{2}$, and $x > 0$, we have $2c - 1 \geq 0$, and $(2c-1)x \geq 0$.
Hence, the numerator satisfies $(2c-1)x + c \geq c \geq \frac{1}{2} > 0$, and the denominator is always positive since is a product of squares, and both squares are non-zero since $x > 0$.
We can now conclude that $g'(x) > 0$ given $c \geq \frac{1}{2}$ for $x > 0$, as desired.

(b) If $0 \leq c < \frac{1}{2}$, $g'(x) < 0$ if and only if

$$\begin{aligned} (2c-1)x + c &< 0 \\ (1-2c)x - c &> 0 \\ (1-2c)x &> c \\ x &> \frac{c}{1-2c}, \end{aligned}$$

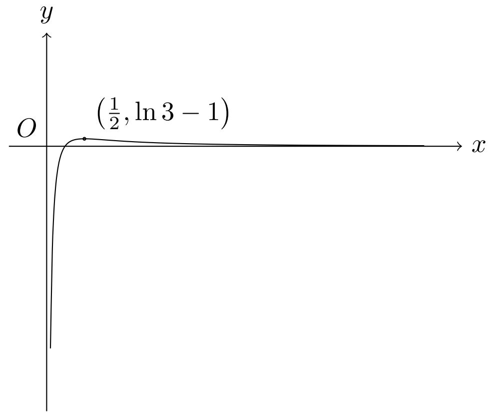
and the values of x are $x > \frac{c}{1-2c}$.

2. (a) If $c = \frac{3}{4} \geq \frac{1}{2}$, we can see that g is always increasing.
As $x \to \infty$, $\frac{x+c}{x(x+1)} \to 0$, $\ln\left(1 + \frac{1}{x}\right) \to \ln 1 = 0$. Hence, $g(x) \to 0$.
Since g is increasing it must stay entirely below the x-axis.
The sketch is as follows.



(b) If $c = \frac{1}{4} \in \left[0, \frac{1}{2}\right)$, it must be the case that $g'(x) > 0$ for $0 < x < \frac{c}{1-2c} = \frac{1}{2}$, and $g'(x) < 0$ for $x > \frac{1}{2}$.

Hence, $x = \frac{1}{2}$ is a maximum on the graph, and the corresponding y-coordinate is $g\left(\frac{1}{2}\right) = \ln 3 - 1$.
Similarly, as $x \to \infty$, $g(x) \to 0$.
The sketch is as follows.



3. We have

$$\begin{aligned}
f(x) &= \left(1 + \frac{1}{x}\right)^{x+c} \\
\ln f(x) &= (x + c) \ln \left(1 + \frac{1}{x}\right) \\
\frac{f'(x)}{f(x)} &= \ln (1 + x) - (x + c) \frac{1}{x(x+1)} \\
\frac{f'(x)}{f(x)} &= g(x) \\
f'(x) &= f(x)g(x).
\end{aligned}$$

$f(x)$ is positive for $x > 0$, and hence $f'(x)$ takes the same sign as $g(x)$.

(a) If $c \geq \frac{1}{2}$, g is increasing and has a limit of 0 at infinity. Hence, $g(x)$ is negative for all $x > 0$, which means $f'(x)$ is negative for all $x > 0$, and hence f is decreasing.

(b) If $0 < c < \frac{1}{2}$, g is negative first, then increases to a positive value, and remains positive and approaches 0 decreasing from above. Hence, f' is first positive and then negative, so f must have a turning point.

(c) If $c = 0$,

$$g'(x) = \frac{-x}{(x+1)^2 x^2} = -\frac{1}{(x+1)^2 x}$$

is always negative, and $\lim_{x \to 0^+} g'(x) = -\infty$, $\lim_{x \to \infty} g'(x) = 0$.
We have

$$g(x) = \ln \left(1 + \frac{1}{x}\right) - \frac{1}{x+1}.$$

As $x \to 0^+$, $\frac{1}{x} \to \infty$, so $\ln\left(1 + \frac{1}{x}\right) \to \infty$, and $-\frac{1}{x+1} \to -\frac{1}{1} = -1$. Hence, $g(x) \to \infty$.
As $x \to \infty$, $g(x) \to 0$.
Since g is decreasing, it must be the case that g is always positive.
This means that f' is always positive as well, and hence f is increasing.