2024.3 Question 2

1. (a) We have

$$\sqrt{4x^2-8x+64} \le |x+8| \iff 0 \le 4x^2-8x+64 \le (x+8)^2.$$

The left inequality can be simplified as follows:

$$\begin{aligned} 4x^2-8x+64 &\ge 0 \\ x^2-2x+16 &\ge 0 \\ (X-1)^2+15 &\ge 0, \end{aligned}$$

which is always true.
The right inequality can be simplified as follows:

$$\begin{aligned} 4x^2-8x+64 &\le (x+8)^2 \\ 4x^2-8x+64 &\le x^2+16x+64 \\ 3x^2-24x &\le 0 \\ x(x-8) &\le 0, \end{aligned}$$

which solves to $0 \le x \le 8$.
Hence, the solution to the original inequality is $x \in [0,8]$.

(b) WE have

$$\sqrt{4x^2-8x+64} \le |3x-8| \iff 0 \le 4x^2-8x+64 \le (3x-8)^2.$$

The left inequality is always true from the previous part.
The right inequality can be simplified as follows:

$$\begin{aligned} 4x^2-8x+64 &\le (3x-8)^2 \\ 4x^2-8x+64 &\le 9x^2-48x+64 \\ 5x^2-40x &\ge 0 \\ x(x-8) &\ge 0, \end{aligned}$$

which solves to $x \le 0$ or $x \ge 8$.
Hence, the solution to the original inequality is $x \in (-\infty,0] \cup [8,\infty)$.

2. (a) We have

$$\begin{aligned} \left(\sqrt{4x^2-8x+64}+2(x-1)\right) f(x) &= \left(\sqrt{4x^2-8x+64}\right)^2 - [2(x-1)]^2 \\ &= \left(4x^2-8x+64\right) - 4\left(x^2-2x+1\right) \\ &= \left(4x^2-8x+64\right) - \left(4x^2-8x+4\right) \\ &= 60. \end{aligned}$$

Hence,

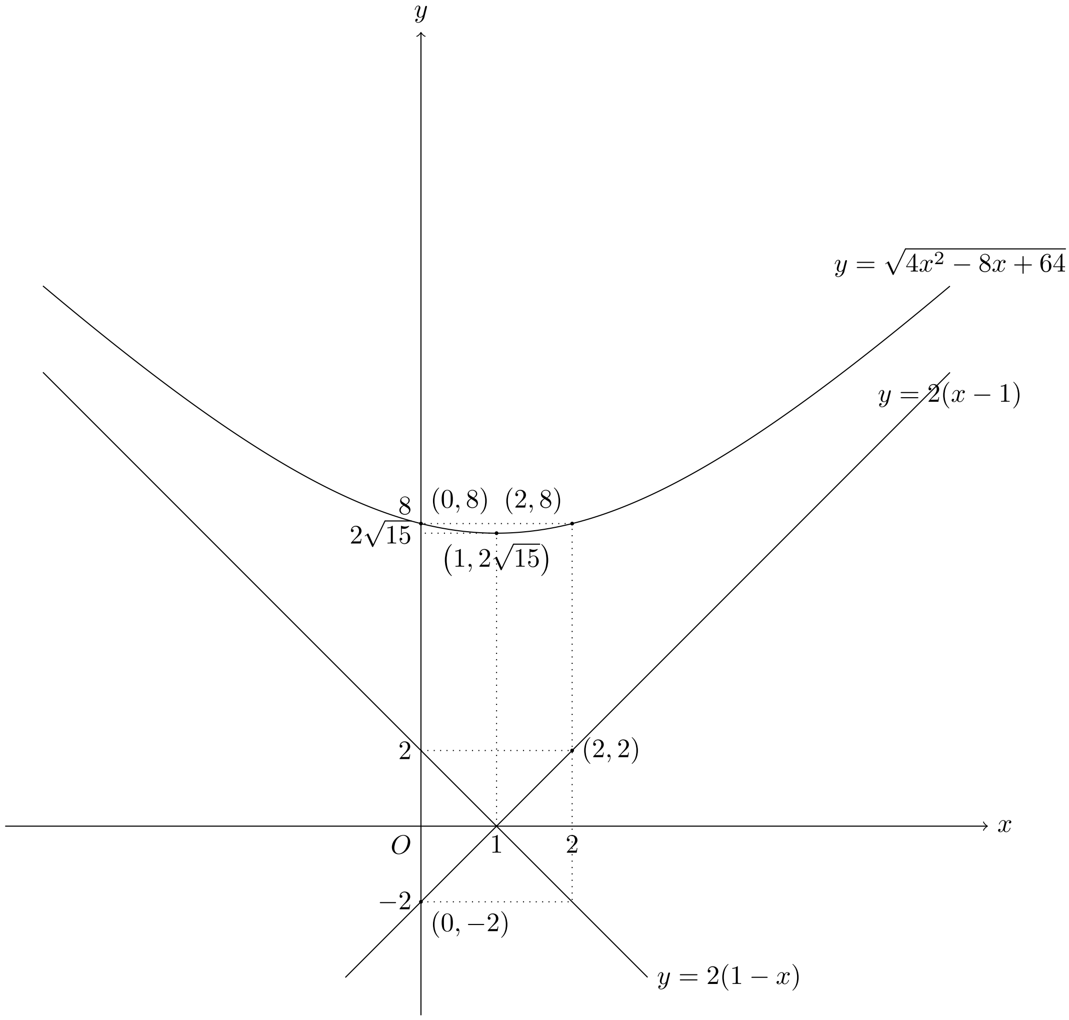
$$f(x) = \frac{60}{\sqrt{4x^2-8x+64}+2(x-1)}.$$

As $x \to \infty$, $\sqrt{4x^2-8x+64} \to \infty$, $2(x-1) \to \infty$.
Hence, $f(x) \to 0$ as $x \to \infty$.

(b) Let $f_1(x) = \sqrt{4x^2-8x+64}$, $f_2(x) = 2(x-1)$.
$f_1(0) = \sqrt{64} = 8$, and $f_2(0) = 2 \cdot (-1) = -2$.
We have $f(x) = f_1(x) - f_2(x) > 0$ from the previous part, and that $f_1(x)$ is defined for all x and is always positive.
Furthermore,

$$f_1(x) = 2\sqrt{x^2-2x+16} = 2\sqrt{(x-1)^2+15},$$

and hence f_1 decreases on $(-\infty,1)$ and increases on $(1,\infty)$, taking a minimum of $f_1(1) = 2\sqrt{15}$.
In terms of symmetry, we have $f_1(1-x) = f_1(1+x)$ and $f_2(1-x) = -f_2(1+x)$. f_2 is an asymptote to f_1 as $x \to \infty$, and $-f_2$ is an asymptote to f_1 as $x \to -\infty$.
Hence, the sketch looks as follows.



3. Let $x = 3$, and we must have $\sqrt{4 \cdot 9 - 5 \cdot 3 + 4} = |3m + c|$, and hence $5 = |3m + c|$.

   This is only achievable for $m = \pm 2$ due to the diagram – the solution set can only be 'one-sided' if on the other side the absolute value is eventually 'parallel' to the curve.

   We let $m = 2$, and hence $5 = |6 + c|$, which gives $c = -1$ or $c = -11$.

   We would like to show that the desired value is $c = -1$, and that $c = -11$ does not work.

$$\sqrt{4x^2 - 5x + 4} \leq |2x - 1| \iff 0 \leq 4x^2 - 5x + 4 \leq (2x - 1)^2.$$

   The left inequality can be simplified as

$$0 \leq 4x^2 - 5x + 4 = \left(2x - \frac{5}{4}\right)^2 + \frac{39}{16},$$

   and hence is always true.

   The right inequality can be simplified as

$$\begin{aligned} 4x^2 - 5x + 4 &\leq (2x - 1)^2 \\ 4x^2 - 5x + 4 &\leq 4x^2 - 4x + 1 \\ x &\geq 3, \end{aligned}$$

   and hence the solution set to the whole inequality is $x \geq 3$ as desired.

   On the other hand, for the case of $c = -11$, we have

$$\sqrt{4x^2 - 5x + 4} \leq |2x - 11| \iff 0 \leq 4x^2 - 5x + 4 \leq (2x - 11)^2,$$

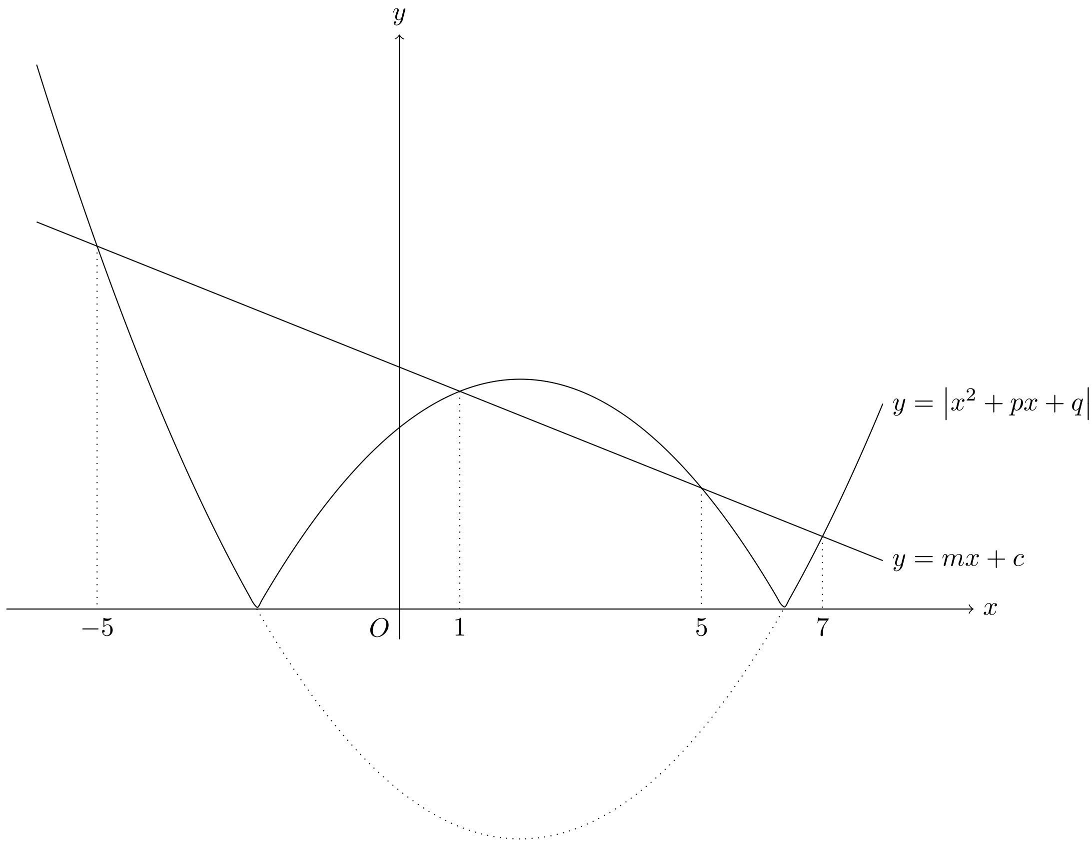
and the left inequality is always true by previously. However, the right inequality simplifies as

$$\begin{aligned} 4x^2 - 5x + 4 &\leq (2x - 11)^2 \\ 4x^2 - 5x + 4 &\leq 4x^2 - 44x + 121 \\ 39x &\leq 117 \\ x &\leq 3, \end{aligned}$$

and the inequality is in the wrong direction.

Hence, a possible value of m is 2, and the corresponding value of c is -1.

4. The diagram as follows shows the only possibility of the configuration.



Hence, we must have $x^2 + px + q = mx + c$ for $x = -5$ and $x = 7$, and $x^2 + px + q = -mx - c$ for $x = 1$ and $x = 5$.

$$\begin{cases} 25 - 5p + q = -5m + c, \\ 49 + 7p + q = 7m + c, \\ \quad 1 + p + q = -(m + c), \\ 25 + 5p + q = -(5m + c). \end{cases}$$

Subtracting the first equation from the final equation gives $10p = -2c$, and hence $c = -5p$.

Subtracting the first equation from the second equation gives us $24 + 12p = 12m$, and hence $m = 2 + p$.

Putting these into the third equation gives

$$\begin{aligned} q &= -m - c -; -1 \\ &= -(2 + p) - (-5p) - p - 1 \\ &= 3p - 3. \end{aligned}$$

Putting all these into the final equation gives

$$\begin{aligned}25 + 5p + (3p - 3) &= - [5(2 + p) + (-5p)] \\ 25 + 8p - 3 &= -(10 + 5p - 5p) \\ 22 + 8p &= -10 \\ 8p &= -32 \\ p &= -4,\end{aligned}$$

and so $q = -15, m = -2, c = 20$. Hence,

$$(p, q, m, c) = (-4, -15, -2, 20).$$