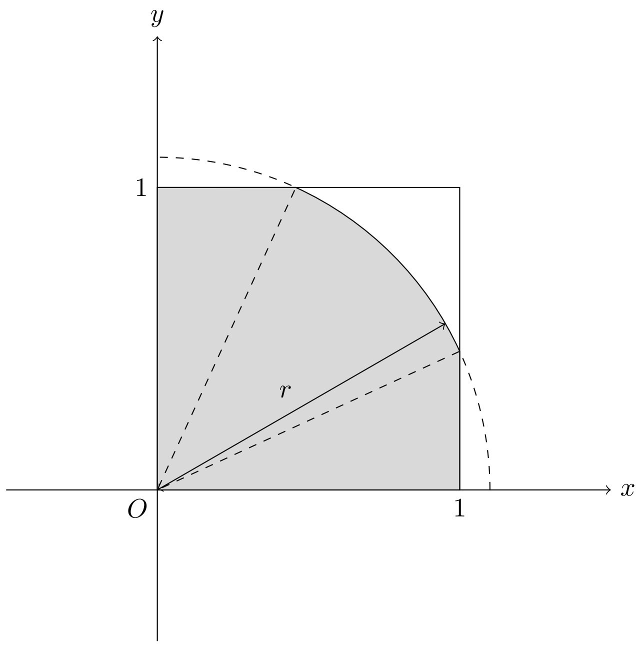
2024.3 Question 12

1. For $1 \leq r \leq \sqrt{2}$, the diagram looks as follows.



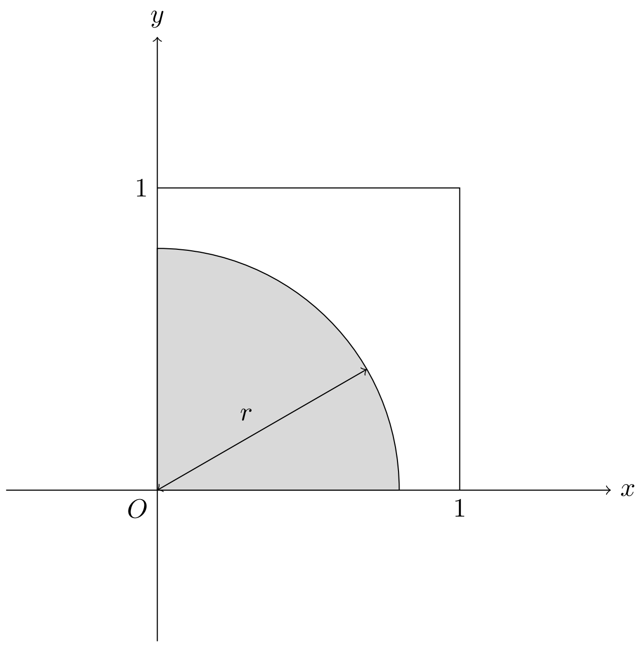
The angle between the (shallower) radius which just intersects the square and x axis is given by $\arccos \frac{1}{r}$, and so is the one steeper and the y-axis.

Hence, the cumulative distribution function is given by

$$\begin{aligned} \mathrm{P}(R \leq r) &= \frac{\text{shaded area}}{1^2} \\ &= \text{shaded area} \\ &= \frac{1}{2} \cdot r^2 \cdot \left(\frac{\pi}{2} - 2\arccos\frac{1}{r}\right) + 2 \cdot \frac{1}{2} \cdot 1 \cdot \sqrt{r^2 - 1} \\ &= \sqrt{r^2 - 1} + \frac{\pi r^2}{4} - r^2 \arccos\frac{1}{r}, \end{aligned}$$

as desired.

For $0 \leq r \leq 1$, the diagram is as follows.



Hence,

$$P(R \leq r) = \text{shaded area} = \frac{\pi r^2}{4}.$$

Hence, the cumulative distribution function is given by

$$P(R \leq r) = \begin{cases} 0, & r < 0, \\ \frac{\pi r^2}{4}, & 0 \leq r < 1, \\ \sqrt{r^2 - 1} + \frac{\pi r^2}{4} - r^2 \arccos \frac{1}{r}, & 1 \leq r < 2, \\ 1, & 2 \leq r. \end{cases}$$

2. Let f be the probability density function of R. Hence, by differentiating, for $0 \leq r \leq \sqrt{2}$, it is given by

$$\begin{aligned} f(r) &= \frac{\mathrm{d}}{\mathrm{d}r} P(R \leq r) \\ &= \begin{cases} \frac{\pi r}{2}, & 0 \leq r \leq 1, \\ \frac{r}{\sqrt{r^2-1}} + \frac{\pi r}{2} - 2r \arccos \frac{1}{r} - \frac{1}{\sqrt{1-\left(\frac{1}{r}\right)^2}}, & 1 \leq r \leq \sqrt{2}, \end{cases} \\ &= \begin{cases} \frac{\pi r}{2}, & 0 \leq r \leq 1, \\ \frac{\pi r}{2} - 2r \arccos \frac{1}{r}, & 1 \leq r \leq \sqrt{2}. \end{cases} \end{aligned}$$

Hence, the expectation is given by

$$\begin{aligned}
\mathrm{E}(R) &= \int_0^1 r \cdot \frac{\pi r}{2} \,\mathrm{d}r + \int_1^{\sqrt{2}} r \cdot \left[\frac{\pi r}{2} - 2r \arccos \frac{1}{r}\right] \mathrm{d}r \\
&= \int_0^{\sqrt{2}} \frac{\pi r^2}{2} \,\mathrm{d}r - 2 \int_1^{\sqrt{2}} r^2 \arccos \frac{1}{r} \,\mathrm{d}r \\
&= \left[\frac{\pi r^3}{6}\right]_0^{\sqrt{2}} - \frac{2}{3} \int_1^{\sqrt{2}} \arccos \frac{1}{r} \,\mathrm{d}r^3 \\
&= \frac{2\sqrt{2}\pi}{6} - \frac{2}{3} \left[\arccos \frac{1}{r} \cdot r^3\right]_1^{\sqrt{2}} + \frac{2}{3} \int_1^{\sqrt{2}} r^3 \,\mathrm{d} \arccos \frac{1}{r} \\
&= \frac{\sqrt{2}\pi}{3} - \frac{2}{3} \cdot \arccos \frac{1}{\sqrt{2}} \cdot 2\sqrt{2} + \frac{2}{3} \cdot \arccos 1 \cdot 1 + \frac{2}{3} \cdot \int_1^{\sqrt{2}} r^3 \cdot \left(-\frac{1}{r^2}\right) \cdot \left(-\frac{1}{\sqrt{1 - \left(\frac{1}{r}\right)^2}}\right) \mathrm{d}r \\
&= \frac{\sqrt{2}\pi}{3} - \frac{2}{3} \cdot \frac{\pi}{4} \cdot 2\sqrt{2} + \frac{2}{3} \int_1^{\sqrt{2}} r \cdot \frac{r}{\sqrt{r^2 - 1}} \,\mathrm{d}r \\
&= \frac{\sqrt{2}\pi}{3} - \frac{\sqrt{2}\pi}{3} + \frac{2}{3} \int_1^{\sqrt{2}} \frac{r^2}{\sqrt{r^2 - 1}} \,\mathrm{d}r \\
&= \frac{2}{3} \int_1^{\sqrt{2}} \frac{r^2}{\sqrt{r^2 - 1}} \,\mathrm{d}r,
\end{aligned}$$

as desired.

3. To integrate this, we let $r = \cosh t$, and hence $\frac{\mathrm{d}r}{\mathrm{d}t} = \sinh t$. When $r = 1$, $t = 0$. When $r = \sqrt{2}$, $t = \ln\left(\sqrt{2} + \sqrt{\sqrt{2}^2 - 1}\right) = \ln(\sqrt{2} + 1)$.

   Hence,

$$\begin{aligned}
\mathrm{E}(R) &= \frac{2}{3} \int_1^{\sqrt{2}} \frac{r^2}{\sqrt{r^2 - 1}} \,\mathrm{d}r \\
&= \frac{2}{3} \int_0^{\ln(\sqrt{2}+1)} \frac{\cosh^2 t}{\sinh t} \cdot \sinh t \,\mathrm{d}t \\
&= \frac{2}{3} \int_0^{\ln(\sqrt{2}+1)} \cosh^2 t \,\mathrm{d}t \\
&= \frac{2}{3} \int_0^{\ln(\sqrt{2}+1)} \frac{e^{2t} + e^{-2t} + 2}{4} \,\mathrm{d}t \\
&= \frac{1}{2} \left[e^{2t} - e^{-2t}\right]_0^{\ln(\sqrt{2}+1)} + \frac{1}{3} [t]_0^{\ln(\sqrt{2}+1)} \\
&= \frac{1}{12} \cdot \left[(\sqrt{2}+1)^2 - (\sqrt{2}+1)^{-2} - e^{2\cdot 0} + e^{-2\cdot 0}\right] + \frac{1}{3} \cdot \left(\ln(\sqrt{2}+1) - 0\right) \\
&= \frac{1}{2} \left[2 + 1 + 2\sqrt{2} - (\sqrt{2} - 1)^2\right] + \frac{1}{3} \ln(\sqrt{2}+1) \\
&= \frac{1}{2} \cdot 4\sqrt{2} + \frac{1}{3} \ln(\sqrt{2}+1) \\
&= \frac{1}{3} \left(\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right),
\end{aligned}$$

   as desired.