2024.3 Question 11

1. We notice that

$$\begin{aligned}\text{LHS} &= r\binom{2n}{r} \\ &= r \cdot \frac{(2n)!}{r!(2n-r)!} \\ &= \frac{(2n)!}{(r-1)!(2n-r)!},\end{aligned}$$

   and

$$\begin{aligned}\text{RHS} &= (2n+1-r)\binom{2n}{2n+1-r} \\ &= (2n+1-r) \cdot \frac{(2n)!}{(r-1)!(2n+1-r)!} \\ &= \frac{(2n)!}{(r-1)!(2n-r)!}.\end{aligned}$$

   Hence,

$$r\binom{2n}{r} = (2n+1-r)\binom{2n}{2n+1-r}$$

   as desired.

   Summing this from $r = n+1$ to $2n$, we have

$$\begin{aligned}\sum_{r=n+1}^{2n} r\binom{2n}{r} &= \sum_{r=n+1}^{2n} (2n+1-r)\binom{2n}{2n+1-r} \\ &= \sum_{r=1}^{n} (2n+1-(2n+1-r))\binom{2n}{2n+1-(2n+1-r)} \\ &= \sum_{r=1}^{n} r\binom{2n}{r},\end{aligned}$$

   and hence

$$\begin{aligned}\sum_{r=0}^{2n} r\binom{2n}{r} &= \sum_{r=1}^{2n} r\binom{2n}{r} \\ &= \sum_{r=1}^{n} r\binom{2n}{r} + \sum_{r=n+1}^{2n} r\binom{2n}{r} \\ &= \sum_{r=n+1}^{2n} r\binom{2n}{r} + \sum_{r=n+1}^{2n} r\binom{2n}{r} \\ &= 2\sum_{r=n+1}^{2n} r\binom{2n}{r},\end{aligned}$$

   as desired.

2. For $n+1 \leq x \leq 2n$, we have

$$\mathrm{P}(X = x) = 2 \cdot \frac{\binom{2n}{x}}{2^{2n}}.$$

   For $x = n$, we have

$$\mathrm{P}(X = x) = \frac{\binom{2n}{n}}{2^{2n}}.$$

We have $n \leq X \leq 2n$, and hence

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{x=n}^{2n} x \, \mathrm{P}(X = x) \\
&= \frac{n\binom{2n}{n}}{2^{2n}} + \frac{2}{2^{2n}} \sum_{x=n+1}^{2n} x \binom{2n}{x} \\
&= \frac{n\binom{2n}{n}}{2^{2n}} + 2^{-2n} \sum_{r=0}^{2n} r \binom{2n}{r} \\
&= \frac{n\binom{2n}{n}}{2^{2n}} + 2^{-2n}(2n)2^{2n-1} \\
&= n + \frac{n\binom{2n}{n}}{2^{2n}} \\
&= n\left(1 + \frac{1}{2^{2n}}\binom{2n}{n}\right)
\end{aligned}$$

as desired.

3. First, we have that

$$\frac{1}{2^{2n}}\binom{2n}{n} > 0$$

for all positive integers n.

Taking the ratio of two consecutive terms, we have

$$\begin{aligned}
\frac{\frac{1}{2^{2n}}\binom{2n}{n}}{\frac{1}{2^{2(n+1)}}\binom{2(n+1)}{n+1}} &= \frac{2^{2n+2}\frac{(2n)!}{n!n!}}{2^{2n}\frac{(2n+2)!}{(n+1)!(n+1)!}} \\
&= 4 \cdot \frac{(n+1)^2}{(2n+2)(2n+1)}.
\end{aligned}$$

We have that the following are equivalent:

$$\begin{aligned}
\frac{1}{2^{2n}}\binom{2n}{n} &> \frac{1}{2^{2(n+1)}}\binom{2(n+1)}{n+1} \\
\frac{\frac{1}{2^{2n}}\binom{2n}{n}}{\frac{1}{2^{2(n+1)}}\binom{2(n+1)}{n+1}} &> 1 \\
\frac{4(n+1)^2}{(2n+2)(2n+1)} &> 1 \\
4n^2 + 8n + 4 &> 4n^2 + 6n + 2 \\
2n + 2 &> 0
\end{aligned}$$

and this obviously true for all positive integers n.

This means that $\frac{1}{2^{2n}}\binom{2n}{n}$ decreases as n increases.

4. The winning is given by $X - n$, and hence the expected winnings per pound is $\frac{1}{2^{2n}}\binom{2n}{n}$. This is maximised when $n = 1$ which gives a value of $\frac{1}{2}$.