2024.3 Question 1

1. For the first identity, notice that

$$\begin{aligned}\frac{1}{r+1}-\frac{1}{r}+\frac{1}{r^2} &= \frac{r^2-r(r+1)+(r+1)}{r^2(r+1)}\\ &= \frac{r^2-r^2-r+r+1}{r^2(r+1)}\\ &= \frac{1}{r^2(r+1)},\end{aligned}$$

and hence using this,

$$\begin{aligned}\sum_{r=1}^{N}\frac{1}{r^2(r+1)} &= \sum_{r=1}^{N}\left(\frac{1}{r+1}-\frac{1}{r}+\frac{1}{r^2}\right)\\ &= \sum_{r=1}^{N}\frac{1}{r^2}+\sum_{r=1}^{N}\frac{1}{r+1}-\sum_{r=1}^{N}\frac{1}{r}\\ &= \sum_{r=1}^{N}\frac{1}{r^2}+\sum_{r=2}^{N+1}\frac{1}{r}-\sum_{r=1}^{N}\frac{1}{r}\\ &= \sum_{r=1}^{N}\frac{1}{r^2}-\frac{1}{1}+\frac{1}{N+1}\\ &= \sum_{r=1}^{N}\frac{1}{r^2}-1+\frac{1}{N+1}.\end{aligned}$$

Let $N\to\infty$, and we have $\frac{1}{N+1}\to 0$, and hence

$$\sum_{r=1}^{\infty}\frac{1}{r^2(r+1)}=\sum_{r=1}^{\infty}\frac{1}{r^2}-1=\frac{\pi^2}{6}-1.$$

2. By partial fractions, let

$$\frac{1}{r^2(r+1)(r+2)}=\frac{Ar+B}{r^2}+\frac{C}{r+1}+\frac{D}{r+2}$$

for real constants A, B, C and D.

Hence,

$$(Ar+B)(r+1)(r+2)+Cr^2(r+2)+Dr^2(r+1)=1.$$

Let $r=-2$, we have $D\cdot(-2)^2\cdot(-1)=-4D=1$, and hence $D=-\frac{1}{4}$.
Let $r=-1$, we have $C\cdot(-1)^2\cdot 1=C=1$, and hence $C=1$.
Let $r=0$, we have $B\cdot 1\cdot 2=1$, and hence $B=\frac{1}{2}$.
Considering the coefficient of r^3, we have $A+C+D=0$, and hence $A=-\frac{3}{4}$.
Hence,

$$\frac{1}{r^2(r+1)(r+2)}=-\frac{3}{4}\cdot\frac{1}{r}+\frac{1}{2}\cdot\frac{1}{r^2}+\frac{1}{r+1}-\frac{1}{4}\cdot\frac{1}{r+2}.$$

Therefore,

$$\begin{aligned}\sum_{r=1}^{N}\frac{1}{r^2(r+1)(r+2)} &= -\frac{3}{4}\sum_{r=1}^{N}\frac{1}{r}+\frac{1}{2}\sum_{r=1}^{N}\frac{1}{r^2}+\sum_{r=1}^{N}\frac{1}{r+1}-\frac{1}{4}\sum_{r=1}^{N}\frac{1}{r+2}\\ &= \frac{1}{2}S_N-\frac{3}{4}\cdot\sum_{r=1}^{N}\frac{1}{r}+\sum_{r=2}^{N+1}\frac{1}{r}-\frac{1}{4}\sum_{r=3}^{N+2}\frac{1}{r}\\ &= \frac{1}{2}S_N-\frac{3}{4}\sum_{r=3}^{N}\frac{1}{r}+\sum_{r=3}^{N}\frac{1}{r}-\frac{1}{4}\sum_{r=3}^{N}\frac{1}{r}\\ &= \frac{1}{2}S_N-\frac{3}{4}\left(\frac{1}{1}+\frac{1}{2}\right)+\left(\frac{1}{2}+\frac{1}{N+1}\right)-\frac{1}{4}\left(\frac{1}{N+1}+\frac{1}{N+2}\right)\\ &= \frac{1}{2}S_N-\frac{9}{8}+\frac{4}{8}+\frac{3}{4}\cdot\frac{1}{N+1}-\frac{1}{4}\cdot\frac{1}{N+2}\\ &= \frac{1}{2}S_N-\frac{5}{8}+\frac{3}{4}\cdot\frac{1}{N+1}-\frac{1}{4}\cdot\frac{1}{N+2}.\end{aligned}$$

Let $N\to\infty$, we have $\frac{1}{N+1},\frac{1}{N+2}\to 0$, and hence

$$\sum_{r=1}^{\infty}\frac{1}{r^2(r+1)(r+2)}=\frac{1}{2}\lim_{N\to\infty}S_N-\frac{5}{8}=\frac{\pi^2}{12}-\frac{5}{8}.$$

3. Similarly, let

$$\frac{1}{r^2(r+1)^2}=\frac{A}{r^2}+\frac{B}{r}+\frac{C}{(r+1)^2}+\frac{D}{r+1}$$

for some real constants A,B,C and D.

Hence,

$$1=A(r+1)^2+Br(r+1)^2+Cr^2+Dr^2(r+1).$$

Let $r=0$, and we have $A=1$. Let $r=-1$, and we have $C=1$. Considering the coefficient of r^3 we have $B+D=0$, and for r, $2A+B=0$.

Hence, $B=-2,D=2$, and

$$\frac{1}{r^2(r+1)^2}=\frac{1}{r^2}-\frac{2}{r}+\frac{1}{(r+1)^2}+\frac{2}{r+1}.$$

Therefore, for natural numbers N, we have

$$\begin{aligned}\sum_{r=1}^{N}\frac{1}{r^2(r+1)^2} &= \sum_{r=1}^{N}\frac{1}{r^2}+\sum_{r=1}^{N}\frac{1}{(r+1)^2}+2\sum_{r=1}^{N}\frac{1}{r+1}-2\sum_{r=1}^{N}\frac{1}{r}\\ &= S_N+\sum_{r=1}^{N+1}\frac{1}{r^2}+2\sum_{r=2}^{N+1}\frac{1}{r}-2\sum_{r=1}^{N}\frac{1}{r}\\ &= S_N+S_{N+1}-\frac{1}{1^2}+2\cdot\frac{1}{N+1}-2\cdot 1\\ &= S_N+s_{N+1}+2\cdot\frac{1}{N+1}-3.\end{aligned}$$

Let $N \to \infty$. $S_N, S_{N+1} \to \frac{\pi^2}{6}$, and $\frac{1}{N+1} \to 0$. Hence,

$$\begin{aligned}\sum_{r=1}^{\infty} \frac{1}{r^2(r+1)^2} &= 2 \cdot \frac{\pi^2}{6} - 3 \\ &= \frac{\pi^2}{3} - 3 \\ &= 2 \cdot \left(\frac{\pi^2}{6} - 1\right) - 1 \\ &= 2\sum_{r=1}^{\infty} \frac{1}{r^2(r+1)} - 1 \\ &= \sum_{r=1}^{\infty} \frac{2}{r^2(r+1)} - 1,\end{aligned}$$

as desired.