2024.2 Question 8

1. Notice that by expanding this square,

$$\begin{aligned}(\sqrt{x_n} - \sqrt{y_n})^2 &= x_n + y_n - 2\sqrt{x_n y_n} \\ &= 2a(x_n, y_n) - 2g(x_n, y_n) \\ &= 2(x_{n+1} - y_{n+1}).\end{aligned}$$

Since this is a square, it must be non-negative, with the equal sign taking if and only if $\sqrt{x_n} = \sqrt{y_n}$, which holds if and only if $x_n = y_n$.

So $x_{n+1} \geq y_{n+1}$, and $x_{n+1} = y_{n+1}$ if and only if $x_n = y_n$.

Since $y_0 < x_0$, we have $y_0 \neq x_0$, and hence $y_1 \neq x_1$. By induction, this shows that $y_n \neq_n$ for all n, and hence for all $n \geq 0$, $y_n < x_n$.

Furthermore,

$$\begin{aligned}x_n - x_{n+1} &= x_n - a(x_n, y_n) \\ &= x_n - \frac{x_n + y_n}{2} \\ &= \frac{x_n - y_n}{2} \\ &> 0,\end{aligned}$$

since $x_n > y_n$ and hence $x_n > x_{n+1}$.

Similarly,

$$\begin{aligned}y_{n+1} - y_n &= g(x_n, y_n) - y_N \\ &= \sqrt{x_n y_n} - y_N \\ &= \sqrt{y_n}(\sqrt{x_n} - \sqrt{y_n}) \\ &> 0,\end{aligned}$$

since $x_n > y_n$ implies $\sqrt{x_n} > \sqrt{y_n}$, and hence $y_n < y_{n+1}$.

Hence, for all $n \in \mathbb{N}$,

$$y_n < x_n < x_{n-1} < x_{n-2} < \cdots < x_0,$$

and $y_{n+1} > y_n$.

Hence, $\{y_N\}_{n=0}^{\infty}$ is an increasing sequence, and is bounded above by x_0.

So there exists $M \in \mathbb{R}$ such that

$$\lim_{n\to\infty} y_n = M.$$

As for the inequality, the left inequality sign is equivalent to $y_{n+1} < x_{n+1}$ which was shown above.

To show the right inequality sign, this is equivalent to showing

$$\begin{aligned}\frac{1}{2}(\sqrt{x_n} - \sqrt{y_n})^2 &< \frac{1}{2}(x_n - y_n) \\ x_n + y_n - 2\sqrt{x_n y_N} &< x_n - y_n \\ 2y_n &< 2\sqrt{x_n y_n} \\ \sqrt{y_n} &< \sqrt{x_n},\end{aligned}$$

which is true since $y_n < x_n$.

Hence,

$$0 < x_{n+1} - y_{n+1} < \frac{1}{2}(x_n - y_n)$$

as desired.

Hence, we have

$$\begin{aligned} 0 &< x_n - y_n \\ &< \frac{1}{2}(x_{n-1} - y_{n-1}) \\ &< \frac{1}{4}(x_{n-2} - y_{n-2}) \\ &< \cdots \\ &< \frac{1}{2^n}(x_0 - y_0), \end{aligned}$$

by induction.

$x_0 - y_0 > 0$ is a positive real constant. Let $n \to \infty$, and by the squeeze theorem, the strict inequalities become weak, and

$$0 \le \lim_{n\to\infty}(x_n - y_n) \le \lim_{n\to\infty}\left(\frac{1}{2^n}(x_0 - y_0)\right) = 0,$$

and hence

$$\lim_{n\to\infty}(x_n - y_n) = 0.$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} x_n &= \lim_{n\to\infty}[(x_n - y_n) + y_n] \\ &= \lim_{n\to\infty}(x_n - y_n) + \lim_{n\to\infty} y_n \\ &= 0 + M \\ &= M, \end{aligned}$$

since both parts of the limit $x_n - y_n$ and y_n exist, the limit of the sum is the sum of the limits of the individual parts.

2. Using this substitution, when $x \to 0^+$, we have $t \to -\infty$, and when $x \to +\infty$, $t \to +\infty$. Also,

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2} + \frac{1}{2} \cdot \frac{pq}{x^2} = \frac{1}{2}\left(1 + \frac{pq}{x^2}\right).$$

Hence, the integral can be simplified as

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\sqrt{\left(\frac{1}{4}(p+q)^2 + t^2\right)(pq + t^2)}} \\ &= \int_0^{\infty} \frac{\frac{1}{2}\left(1 + \frac{pq}{x^2}\right)\mathrm{d}x}{\sqrt{\left(\frac{1}{4}(p+q)^2 + \frac{1}{4}\left(x - \frac{pq}{x}\right)^2\right)\left(pq + \frac{1}{4}\left(x - \frac{pq}{x}\right)^2\right)}} \\ &= \int_0^{\infty} \frac{\frac{1}{2}\left(1 + \frac{pq}{x^2}\right)\mathrm{d}x}{\frac{1}{4}\sqrt{\left(p^2 + 2pq + q^2 + x^2 - 2pq + \frac{p^2q^2}{x^2}\right)\left(4pq + x^2 - 2pq + \frac{p^2q^2}{x^2}\right)}} \\ &= 2\int_0^{\infty} \frac{\left(1 + \frac{pq}{x^2}\right)\mathrm{d}x}{\sqrt{\left(p^2 + q^2 + x^2 + \frac{p^2q^2}{x^2}\right)\left(x^2 + 2pq + \frac{p^2q^2}{x^2}\right)}} \\ &= 2\int_0^{\infty} \frac{(x^2 + pq)\,\mathrm{d}x}{\sqrt{(x^4 + (p^2+q^2)x^2 + p^2q^2)(x^4 + 2pqx^2 + p^2q^2)}} \\ &= 2\int_0^{\infty} \frac{(x^2 + pq)\,\mathrm{d}x}{\sqrt{(x^2+p^2)(x^2+q^2)(x^2+pq)^2}} \\ &= 2\int_0^{\infty} \frac{\mathrm{d}x}{\sqrt{(x^2+p^2)(x^2+q^2)}} \\ &= 2I(p, q), \end{aligned}$$

which means

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\sqrt{\left(\frac{1}{4}(p+q)^2+t^2\right)(pq+t^2)}} = 2I(p,q).$$

But also note that the left-hand side satisfies that

$$\begin{aligned}
\text{LHS} &= \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\sqrt{\left(\frac{1}{4}(p+q)^2+t^2\right)(pq+t^2)}} \\
&= 2\int_0^{\infty} \frac{\mathrm{d}t}{\sqrt{\left[\left(\frac{1}{2}(p+q)\right)^2+t^2\right]\left[(\sqrt{pq})^2+t^2\right]}} \\
&= 2\int_0^{\infty} \frac{\mathrm{d}t}{\sqrt{\left[a(p,q)^2+t^2\right]\left[g(p,q)^2+t^2\right]}} \\
&= 2I(a(p,q),g(p,q)),
\end{aligned}$$

since the integrand is an even function, and so

$$I(p,q) = I(a(p,q),g(p,q)),$$

as desired.

Since $0 < q < p$, let $y_0 = q, x_0 = p$, and hence

$$\begin{aligned}
I(p,q) &= I(x_0,y_0) \\
&= I(a(x_0,y_0),g(x_0,y_0)) \\
&= I(x_1,y_1) \\
&= \cdots \\
&= I(x_n,y_n).
\end{aligned}$$

Let $n \to \infty$, and we have

$$\begin{aligned}
I(p,q) &= I(M,M) \\
&= \int_0^{\infty} \frac{\mathrm{d}x}{M^2+x^2} \\
&= \frac{1}{M}\left[\arctan\left(\frac{x}{M}\right)\right]_0^{\infty} \\
&= \frac{\pi}{2M}.
\end{aligned}$$