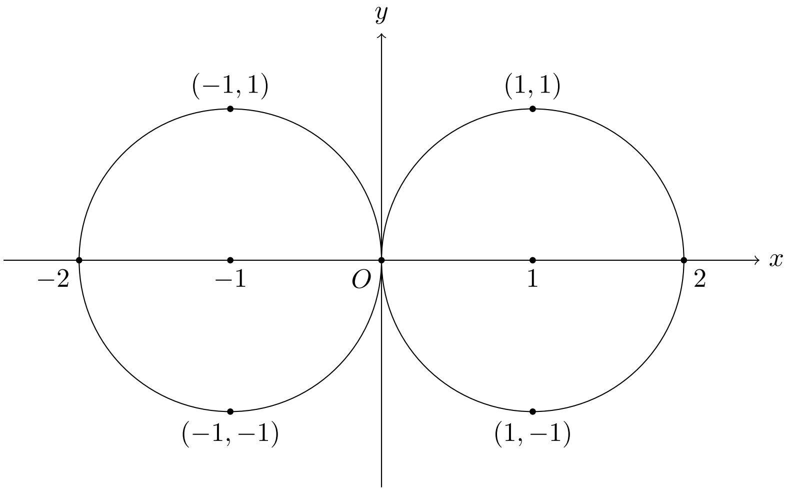
2024.2 Question 7

1. In this case, we have either $y^2 + (x-1)^2 = 1$ (giving a circle with radius 1 centred at $(1,0)$), or $y^2 + (x+1)^2 = 1$ (giving a circle with radius 1 centred at $(-1,0)$).



2. At $y = k$, we have

$$\begin{aligned}[(x-1)^2 + (k^2-1)][(x+1)^2 + (k^2-1)] &= \frac{1}{16} \\ (x-1)^2(x+1)^2 + (k^2-1)[(x-1)^2 + (x+1)^2] + (k^2-1)^2 - \frac{1}{16} &= 0 \\ (x^2-1)^2 + 2(k^2-1)(x^2+1) + (k^4 - 2k^2 + 1) - \frac{1}{16} &= 0 \\ x^4 - 2x^2 + 1 + 2(k^2-1)x^2 + 2(k^2-1) + (k^4 - 2k^2 + 1) - \frac{1}{16} &= 0 \\ x^4 + 2(k^2-2)x^2 + k^4 - \frac{1}{16} &= 0,\end{aligned}$$

as desired.

By completing the square, we have

$$\begin{aligned}(x^2 + (k^2-2))^2 + k^4 - \frac{1}{16} - (k^2-2)^2 &= 0 \\ (x^2 + (k^2-2))^2 &= \frac{65}{16} - 4k^2.\end{aligned}$$

- If $4k^2 > \frac{65}{16}$, i.e. $k^2 > \frac{65}{64}$, the right-hand side is negative, so there will be no intersections.
- If $4k^2 = \frac{65}{16}$, i.e. $k^2 = \frac{65}{64}$, we have

  $$x^2 + (k^2 - 2) = 0,$$

  and hence

  $$x^2 = 2 - k^2 = 2 - \frac{65}{64} = \frac{63}{64},$$

  giving

  $$x = \pm\frac{3\sqrt{7}}{8}.$$

  There will be two intersections.
- If $4k^2 < \frac{65}{16}$, i.e. $k^2 < \frac{65}{64}$, we have

  $$x^2 + (k^2 - 2) = \pm\sqrt{\frac{65}{16} - 4k^2},$$

and hence

$$x^2 = 2 - k^2 \pm \sqrt{\frac{65}{16} - 4k^2}.$$

The case where

$$\begin{aligned} x^2 &= 2 - k^2 + \sqrt{\frac{65}{16} - 4k^2} \\ &> 2 - k^2 \\ &> 2 - \frac{65}{64} \\ &= \frac{63}{64} \\ &> 0 \end{aligned}$$

always gives two solutions for x.

- If $2 - k^2 - \sqrt{\frac{65}{16} - 4k^2} < 0$,

$$\begin{aligned} 2 - k^2 - \sqrt{\frac{65}{16} - 4k^2} &< 0 \\ \sqrt{\frac{65}{16} - 4k^2} &> 2 - k^2 \\ \frac{65}{16} - 4k^2 &> k^4 - 4k^2 + 4 \\ k^4 &< \frac{1}{16} \\ k^2 &< \frac{1}{4}, \end{aligned}$$

  there are no solutions for the case where the minus sign is taken.
- If $2 - k^2 - \sqrt{\frac{65}{16} - 4k^2} = 0$, $k^2 = \frac{1}{4}$, the minus sign produce precisely one solution $x = 0$, giving 3 intersections in total.
- If $2 - k^2 - \sqrt{\frac{65}{16} - 4k^2} < 0$, $k^2 > \frac{1}{4}$, the minus sign will produce two additional roots, hence giving 4 intersections in total.

To summarise, the number of intersections with the line $y = k$ for each positive value of k is

$$\text{number of intersections} = \begin{cases} 0, & k^2 > \frac{65}{64}, k > \frac{\sqrt{65}}{8}, \\ 2, & k^2 = \frac{65}{64}, k = \frac{\sqrt{65}}{8}, \\ 4, & \frac{1}{4} < k^2 < \frac{65}{64}, \frac{1}{2} < k < \frac{\sqrt{65}}{8}, \\ 3, & k^2 = \frac{1}{4}, k = \frac{1}{2}, \\ 2, & k^2 < \frac{1}{4}, 0 < k < \frac{1}{2}. \end{cases}$$

3. When the point on C_2 has the greatest possible y-coordinate, the two points have x-coordinates

$$x = \pm\frac{3\sqrt{7}}{8},$$

and on C_1 has

$$x = \pm 1.$$

Since $3\sqrt{7} = \sqrt{63} < \sqrt{64} = 8$, we must have $\frac{3\sqrt{7}}{8} < 1$, meaning those on C_2 are closer to the y-axis than those on C_1.

4. If both are negative, then the distance from (x, y) to $(1, 0)$ and $(-1, 0)$ are both less than 1. But this is not possible, since the distance from $(1, 0)$ to $(-1, 0)$ is 2, which means the sum of the distances from (x, y) to those points has to be at least 2.

Therefore, since the product of those two terms are positive for C_2, and they cannot be both negative, they must both be positive, and hence the distance from (x, y) to $(1, 0)$ and $(-1, 0)$ are both more than 1, meaning all points on C_2 lies outside the two circles that make up C_1, which shows that C_2 lies entirely outside C_1.

5. C_2 is symmetric about both the x-axis and the y-axis.

   When $x = 0$, $y^4 = \frac{1}{16}$, and hence $y = \pm\frac{1}{2}$.

   When $y = 0$, $x^2 = 2 + \frac{\sqrt{65}}{16}$, and hence $x = \pm\sqrt{2 + \frac{\sqrt{65}}{4}} = \pm\frac{1}{2}\sqrt{8 + \sqrt{65}}$.

   Hence, the graph looks as follows.