2024.2 Question 6

1. We first look at the base case where $n = 0$, and we have

$$\text{RHS} = \frac{1}{2^{2\cdot 0}}\binom{2\cdot 0}{0} = \frac{1}{2^0}\binom{0}{0} = 1,$$

and LHS $= T_0 = 1$. So the desired statement is satisfied for the base case where $n = 0$.

Assume the original statement is true for some $n = k \geq 0$, that

$$T_n = \frac{1}{2^{2n}}\binom{2n}{n}.$$

Consider $n = k + 1$, we have

$$\begin{aligned} T_n &= T_{k+1} \\ &= \frac{2(k+1)-1}{2(k+1)}T_k \\ &= \frac{2k+1}{2(k+1)} \cdot \frac{1}{2^{2k}}\binom{2k}{k} \\ &= \frac{(2k+1)(2k+2)}{2(k+1)2(k+1)} \cdot \frac{1}{2^{2k}}\frac{(2k)!}{(}k!k!) \\ &= \frac{(2k+2)!}{(k+1)!(k+1)!} \cdot \frac{1}{2^{2k+2}} \\ &= \frac{1}{2^{2(k+1)}}\binom{2(k+1)}{k+1}, \end{aligned}$$

which is precisely the statement for $n = k + 1$.

The original statement is true for $n = 0$, and given it holds for some $n = k \geq 0$, it holds for $n = k + 1$. Hence, by the principle of mathematical induction, the statement

$$T_n = \frac{1}{2^{2n}}\binom{2n}{n}$$

holds for all integers $n \geq 0$, as desired.

2. By Newton's binomial theorem, we have

$$(1-x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x)^3 + \cdots,$$

and notice that the negative signs cancels out, and hence

$$a_n = \frac{\prod_{k=1}^{n}\frac{2k-1}{2}}{n!} = \frac{\prod_{k=1}^{n}(2k-1)}{2^n n!}.$$

Hence, we note that

$$\begin{aligned} \frac{a_r}{a_{r-1}} &= \frac{\prod_{k=1}^{r}(2k-1)/(2^r r!)}{\prod_{k=1}^{r-1}(2k-1)/(2^{r-1}(r-1)!)} \\ &= \frac{2r-1}{2r}, \end{aligned}$$

and hence

$$a_r = \frac{2r-1}{2r}a_{r-1}.$$

Note that $a_0 = 1$ as well. The sequence $\{a_n\}_0^\infty$ and $\{T_n\}_0^\infty$ have the same initial term $a_0 = T_0 = 1$, and they have the same same inductive relationship

$$a_n = \frac{2n-1}{2n}a_{n-1}, T_n = \frac{2n-1}{2n}T_{n-1}.$$

This shows they are the same sequence, hence

$$a_n = T_n$$

for all $n = 0, 1, 2, \cdots$.

3. By Newton'w binomial theorem,

$$(1-x)^{-\frac{3}{2}} = 1 + \frac{\left(-\frac{3}{2}\right)(-x)}{1!} + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-x)}{2!} + \cdots,$$

and so

$$b_n = \frac{\prod_{k=1}^{n} \frac{2k+1}{2}}{n!} = \frac{\prod_{k=1}^{n}(2k+1)}{2^n n!}.$$

Notice that

$$\begin{aligned}
\frac{b_n}{a_n} &= \frac{\prod_{k=1}^{n}(2k+1)/(2^n n!)}{\prod_{k=1}^{n}(2k-1)/2^n n!} \\
&= \frac{\prod_{k=1}^{n}(2k+1)}{\prod_{k=1}^{n}(2k-1)} \\
&= \frac{\prod_{k=2}^{n+1}(2k-1)}{\prod_{k=1}^{n}(2k-1)} \\
&= \frac{2(n+1)-1}{2\cdot 1-1} \\
&= 2n+1,
\end{aligned}$$

and so

$$\begin{aligned}
b_n &= (2n+1)a_n \\
&= (2n+1)\cdot\frac{1}{2^{2n}}\cdot\binom{2n}{n} \\
&= \frac{2n+1}{2^{2n}}\binom{2n}{n}.
\end{aligned}$$

4. By the binomial expansion, we have

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots,$$

and we have

$$(1-x)^{-\frac{1}{2}}\cdot(1-x)^{-1} = (1-x)^{-\frac{3}{2}}.$$

For a particular term in the series expansion for $(1-x)^{-\frac{3}{2}}$, say b_n, we must have

$$b_n x^n = \sum_{t=0}^{n} a_t \cdot x^t \cdot 1 \cdot x^{n-t},$$

and hence

$$b_n = \sum_{t=0}^{n} a_t,$$

which gives

$$\frac{2n+1}{2^{2n}}\binom{2n}{n} = \sum_{r=0}^{n}\frac{1}{2^{2r}}\binom{2r}{r},$$

exactly as desired.