2024.2 Question 5

1. We have
$$f_1(n) = n^2 + 6n + 11 = (n + 3)^2,$$
and so
$$f_1(\mathbb{Z}) = \{(n + 3)^2 + 2 \mid n \in \mathbb{Z}\}.$$
But since if $n \in \mathbb{Z}, n + 3 \in \mathbb{Z}$, and if $n + 3 \in \mathbb{Z}, n \in \mathbb{Z}$, so
$$f_1(\mathbb{Z}) = \{(n + 3)^2 + 2 \mid n \in \mathbb{Z}\} = \{n^2 + 2 \mid n \in \mathbb{Z}\}.$$
We have $F_1(\mathbb{Z}) = \{n^2 + 2 \mid n \in \mathbb{Z}\}$, and so $f_1(\mathbb{Z}) = F_1(\mathbb{Z})$, which shows f_1 and F_1 has the same range/

2. We have
$$g_1(n) = n^2 - 2n + 5 = (n - 1)^2 + 4,$$
and so
$$g_1(\mathbb{Z}) = \{(n - 1)^2 + 4 \mid n \in \mathbb{Z}\} = \{n^2 + 4 \mid n \in \mathbb{Z}\}.$$
The quadratic residues modulo 4 are 0 and 1, and so
$$f_1(\mathbb{Z}) \subseteq \{0 + 2, 1 + 2\} = \{2, 3\} \bmod 4,$$
and
$$g_1(\mathbb{Z}) \subseteq \{0 + 4, 1 + 4\} = \{0, 1\} \bmod 4.$$
Under modulo 4, $f_1(\mathbb{Z}) \cap g_1(\mathbb{Z}) \subseteq \{2, 3\} \cap \{0, 1\} = \varnothing$.
Hence, $f_1(\mathbb{Z}) \cap g_1(\mathbb{Z}) = \varnothing$ under modulo 4, and hence $f_1(\mathbb{Z}) \cap g_1(\mathbb{Z}) = \varnothing$.

3. We have
$$f_2(n) = n^2 - 2n - 6 = (n - 1)^2 - 7,$$
and so
$$f_2(\mathbb{Z}) = \{(n - 1)^2 - 7 \mid n \in \mathbb{Z}\} = \{n^2 - 7 \mid n \in \mathbb{Z}\}.$$
Similarly,
$$g_2(n) = n^2 - 4n + 2 = (n - 2)^2 - 2,$$
and so
$$g_2(\mathbb{Z}) = \{(n - 2)^2 - 2 \mid n \in \mathbb{Z}\} = \{n^2 - 2 \mid n \in \mathbb{Z}\}.$$
So for the intersection, if $t \in f_2(\mathbb{Z}) \cap g_2(\mathbb{Z})$, then there exists $n_1, n_2 \in \mathbb{Z}$,
$$t = n_1^2 - 7 = n_2^2 - 2,$$
and hence
$$n_1^2 - n_2^2 = (n_1 + n_2)(n_1 - n_2) = 5.$$
So
$$(n_1 + n_2, n_1 - n_2) = (\pm 1, \pm 5) \text{ or } (\pm 5, \pm 1),$$
and hence
$$(n_1, n_2) = (\pm 3, \mp 2) \text{ or } (\pm 3, \pm 2),$$
which gives
$$t = (\pm 3)^2 - 7 = 2.$$
Therefore,
$$f_2(\mathbb{Z}) \cap g_2(\mathbb{Z}) = \{2\},$$
and 2 is the only integer which lies in the intersection of the range of f_2 and g_2.

4. Since $p, q \in \mathbb{R}$, we must have $p+q, p-q \in \mathbb{R}$ and hence

$$\begin{aligned}(p+q)^2 &= p^2 + 2pq + q^2 \geq 0,\\ (p-q)^2 &= p^2 - 2pq + q^2 \geq 0.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{3}{4}(p+q)^2 + \frac{1}{4}(p-q)^2 &= \frac{3}{4}\left(p^2 + 2pq + q^2\right) + \frac{1}{4}\left(p^2 - 2pq + q^2\right)\\ &= p^2 + pq + q^2\\ &\geq 0,\end{aligned}$$

as desired.

We have

$$f_3(n) = n^3 - 3n^2 + 7n = (n-1)^3 + 4n + 1 = (n-1)^3 + 4(n-1) + 5,$$

and so

$$f_3(\mathbb{Z}) = \{(n-1)^3 + 4(n-1) + 5 \mid n \in \mathbb{Z}\} = \{n^3 + 4n + 5 \mid n \in \mathbb{Z}\}.$$

We have

$$g_3(\mathbb{Z}) = \{n^3 + 4n - 6 \mid n \in \mathbb{Z}\}.$$

So if $t \in f_3(\mathbb{Z}) \cap g_3(\mathbb{Z})$, then there exists $n_1, n_2 \in \mathbb{Z}$ such that

$$t = n_1^3 + 4n_1 + 5 = n_2^3 + 4n_2 - 6.$$

Hence,

$$(n_1^3 - n_2^3) + 4(n_1 - n_2) = (n_1 - n_2)(n_1^2 + n_1 n_2 + n_2^2 + 4) = -11.$$

Since $n_1^2 + n_1 n_2 + n_2^2 \geq 0$ by the lemma in the previous part, we have $n_1^2 + n_1 n_2 + n_2^2 + 4 \geq 4$. But $n_1^2 + n_1 n_2 + n_2^2 + 4 \mid -11$, and so

$$n_1^2 + n_1 n_2 + n_2^2 + 4 = 11, n_1 - n_2 = -1.$$

Putting $n_2 = n_1 + 1$ into the first equation, we have

$$\begin{aligned}n_1^2 + n_1 n_2 + n_2^2 + 4 &= n_1^2 + n_1(n_1+1) + (n_1+1)^2 + 4\\ &= n_1^2 + n_1^2 + n_1 + n_1^2 + 2n_1 + 1 + 4\\ &= 3n_1^2 + 3n_1 + 5\\ &= 11,\end{aligned}$$

and hence

$$3n_1^2 + 3n_1 - 6 = 3(n_1+2)(n_1-1) = 0,$$

which gives $n_1 = -2$ or $n_1 = 1$, and they correspond to $n_2 = -1$ or $n_2 = 2$.

Hence,

$$t = (-1)^3 + 4(-1) - 6 = -1 - 4 - 6 = -11,$$

or

$$t = 2^3 + 4 \cdot 2 - 6 = 8 + 8 - 6 = 10.$$

Hence,

$$f_3(\mathbb{Z}) \cap g_3(\mathbb{Z}) = \{-11, 10\},$$

and the integers that lie in the intersection of the ranges are -11 and 10.