2024.2 Question 4

1. (a) We first show that $\mathbf{b}$ lies in the plane XOY. Since $\mathbf{b}$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$, it must lie in the plane containing $\mathbf{x} = \overrightarrow{OX}$ and $\mathbf{y} = \overrightarrow{OY}$, which is the plane XOY.
Let α be the angle between $\mathbf{b}$ and $\mathbf{x}$, and let β be the angle between $\mathbf{b}$ and $\mathbf{y}$, where $0 \leq \alpha, \beta \leq \pi$.
We have

$$\begin{aligned}\cos\alpha &= \frac{\mathbf{b}\cdot\mathbf{x}}{|\mathbf{b}||\mathbf{x}|} \\ &= \frac{1}{|\mathbf{b}|}\cdot\frac{(|\mathbf{x}|\mathbf{y}+|\mathbf{y}|\mathbf{x})\cdot\mathbf{x}}{|\mathbf{x}|} \\ &= \frac{1}{|\mathbf{b}|}\cdot\frac{|\mathbf{x}|\cdot(\mathbf{x}\cdot\mathbf{y})+|\mathbf{y}|\cdot|\mathbf{x}|^2}{|\mathbf{x}|} \\ &= \frac{1}{|\mathbf{b}|}\cdot(\mathbf{x}\cdot\mathbf{y}+|\mathbf{x}|\cdot|\mathbf{y}|).\end{aligned}$$

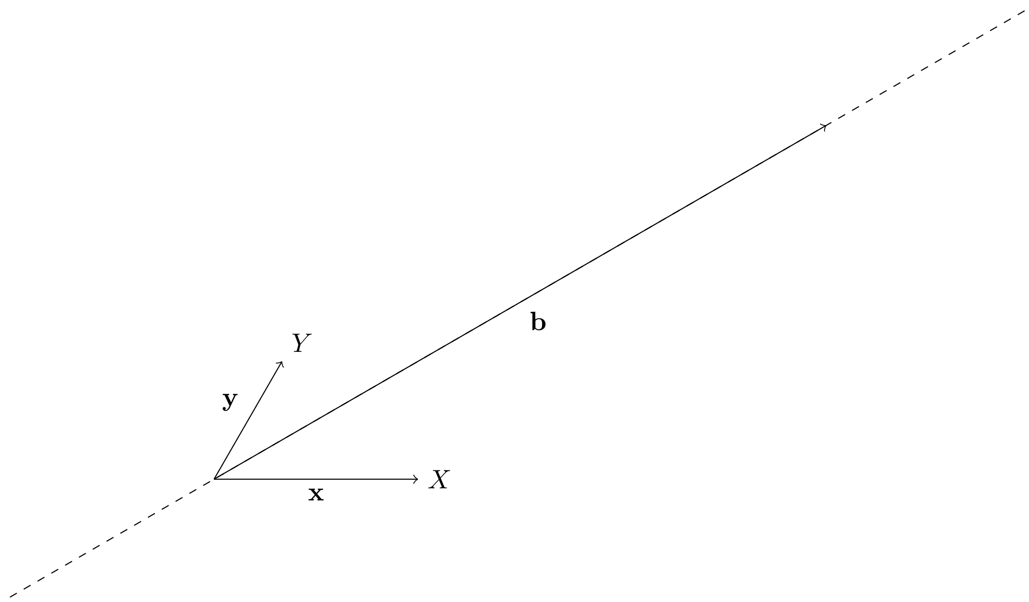
Similarly,

$$\cos\beta = \frac{1}{|\mathbf{b}|}\cdot(\mathbf{x}\cdot\mathbf{y}+|\mathbf{x}|\cdot|\mathbf{y}|) = \cos\alpha.$$

Since the cos function is one-to-one on $[0, \pi]$, we must have $\alpha = \beta$.
Since $\mathbf{x}\cdot\mathbf{y} = |\mathbf{x}|\cdot|\mathbf{y}|\cdot\cos\theta$ where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$, we have $\mathbf{x}\cdot\mathbf{y} \geq -|\mathbf{x}||\mathbf{y}|$, and since $\theta \neq \pi$ (since OXY are non-collinear), we have $\mathbf{x}\cdot\mathbf{y} > -|\mathbf{x}||\mathbf{y}|$, and hence $\cos\alpha = \cos\beta > 0$.
This shows that both angles are less than $\frac{\pi}{2} = 90°$.
Hence, the three conditions

- $\mathbf{b}$ lies in the plane OXY,
- the angle between $\mathbf{b}$ and $\mathbf{x}$ is equal to the angle between $\mathbf{b}$ and $\mathbf{y}$,
- both angles are less than $\frac{\pi}{2} = 90°$

are all satisfied, and we can conclude that $\mathbf{b}$ is a bisecting vector for the plane OXY.



All bisecting vectors must lie on the line containing $\mathbf{b}$ (the dashed line on the diagram), and hence a scalar multiple of $\mathbf{b}$.
Furthermore, since both angles must be less than $\frac{\pi}{2}$, it must not on the opposite as where $\mathbf{b}$ is situated, and hence it must be a positive multiple of $\mathbf{b}$.

(b) If B lies on XY, then $\mathbf{OB} = \mu\mathbf{x} + (1-\mu)\mathbf{y}$ must be a convex combination of $\mathbf{x}$ and $\mathbf{y}$, and hence

$$\lambda\left(|\mathbf{x}|\mathbf{y}+|\mathbf{y}|\mathbf{x}\right) = \mu\mathbf{x}+(1-\mu)\mathbf{y}.$$

Since O, X and Y are not collinear, we must have $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, and hence $\lambda|\mathbf{y}| = \mu$ and $\lambda|\mathbf{x}| = 1 - \mu$, hence giving

$$\lambda = \frac{1}{|\mathbf{x}| + |\mathbf{y}|}$$

We therefore have

$$\begin{aligned}\frac{XB}{BY} &= \frac{\left|\overrightarrow{OB} - \mathbf{x}\right|}{\left|\mathbf{y} - \overrightarrow{OB}\right|} \\ &= \frac{\left|\frac{|\mathbf{x}|}{|\mathbf{x}|+|\mathbf{y}|}\mathbf{y} + \frac{|\mathbf{y}|}{|\mathbf{x}|+|\mathbf{y}|}\mathbf{y}\mathbf{x} - \mathbf{x}\right|}{\left|\frac{|\mathbf{x}|}{|\mathbf{x}|+|\mathbf{y}|}\mathbf{y} + \frac{|\mathbf{y}|}{|\mathbf{x}|+|\mathbf{y}|}\mathbf{y}\mathbf{x} - \mathbf{y}\right|} \\ &= \frac{\left|\frac{|\mathbf{x}|}{|\mathbf{x}|+|\mathbf{y}|}\left(\mathbf{y} - \mathbf{x}\right)\right|}{\left|\frac{|\mathbf{y}|}{|\mathbf{x}|+|\mathbf{y}|}\left(\mathbf{x} - \mathbf{y}\right)\right|} \\ &= \frac{\frac{|\mathbf{x}|}{|\mathbf{x}|+|\mathbf{y}|} \cdot |\mathbf{y} - \mathbf{x}|}{\frac{|\mathbf{y}|}{|\mathbf{x}|+|\mathbf{y}|} \cdot |\mathbf{x} - \mathbf{y}|} \\ &= \frac{|\mathbf{x}|}{|\mathbf{y}|},\end{aligned}$$

which means

$$XB : BY = |\mathbf{x}| : |\mathbf{y}|,$$

which is precisely the angle bisector theorem.

(c) Considering the dot product,

$$\begin{aligned}\overrightarrow{OB} \cdot \overrightarrow{XY} &= \lambda\mathbf{b} \cdot (\mathbf{y} - \mathbf{x}) \\ &= \lambda\left(|\mathbf{x}|\mathbf{y} + |\mathbf{y}|\mathbf{x}\right) \cdot (\mathbf{y} - \mathbf{x}) \\ &= \lambda\left[|\mathbf{x}| \cdot \mathbf{y} \cdot \mathbf{y} + |\mathbf{y}| \cdot \mathbf{x} \cdot \mathbf{y} - |\mathbf{x}| \cdot \mathbf{x} \cdot \mathbf{y} - |\mathbf{y}| \cdot \mathbf{x} \cdot \mathbf{x}\right] \\ &= \lambda\left[|\mathbf{x}| \cdot |\mathbf{y}|^2 + \left[|\mathbf{y}| - |\mathbf{x}|\right]\mathbf{x} \cdot \mathbf{y} - |\mathbf{y}| \cdot |\mathbf{x}|^2\right] \\ &= \lambda\left(|\mathbf{y}| - |\mathbf{x}|\right)\left(|\mathbf{x}||\mathbf{y}| + \mathbf{x} \cdot \mathbf{y}\right) \\ &= 0.\end{aligned}$$

Since O, X, Y are not collinear, $\mathbf{x} \cdot \mathbf{y} > -|\mathbf{x}||\mathbf{y}|$, and hence $|\mathbf{x}||\mathbf{y}| + \mathbf{x} \cdot \mathbf{y} > 0$.
Also, $\lambda = \frac{1}{|\mathbf{x}|+|\mathbf{y}|} \neq 0$.
So it must be the case that $|\mathbf{x}| - |\mathbf{y}| = 0$, which means $|\mathbf{x}| = |\mathbf{y}|$.
Hence, $OX = OY$, and triangle OXY is isosceles.

2. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be the bisecting vectors for QOR, ROP and POQ respectively, and let $\mathbf{p} = \overrightarrow{OP}$, $\mathbf{q} = \overrightarrow{OQ}$, $\mathbf{r} = \overrightarrow{OR}$.

   Let i, j, k be some arbitrary positive real constant.

   From the question, we have

$$\begin{cases}\mathbf{u} = i\left(|\mathbf{q}|\mathbf{r} + |\mathbf{r}|\mathbf{q}\right), \\ \mathbf{v} = j\left(|\mathbf{r}|\mathbf{p} + |\mathbf{p}|\mathbf{r}\right), \\ \mathbf{w} = k\left(|\mathbf{p}|\mathbf{q} + |\mathbf{q}|\mathbf{p}\right).\end{cases}$$

   Considering a pair of dot-product, we have

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= ij \cdot \left(|\mathbf{q}||\mathbf{r}|\mathbf{r} \cdot \mathbf{p} + |\mathbf{p}||\mathbf{q}|\mathbf{r} \cdot \mathbf{r} + |\mathbf{r}||\mathbf{r}|\mathbf{p} \cdot \mathbf{q} + |\mathbf{r}||\mathbf{p}|\mathbf{q} \cdot \mathbf{r}\right) \\ &= ij|\mathbf{r}|\left(|\mathbf{q}|\mathbf{r} \cdot \mathbf{q} + |\mathbf{p}|\mathbf{r} \cdot \mathbf{q} + |\mathbf{p}||\mathbf{q}||\mathbf{r}| + |\mathbf{r}|\mathbf{p} \cdot \mathbf{q}\right) \\ &= ij|\mathbf{r}|^2|\mathbf{p}||\mathbf{q}|\left(\cos\langle\mathbf{p}, \mathbf{r}\rangle + \cos\langle\mathbf{r}, \mathbf{q}\rangle + \cos\langle\mathbf{p}, \mathbf{q}\rangle + 1\right),\end{aligned}$$

where $\langle \mathbf{a}, \mathbf{b} \rangle$ denotes the angle between $\mathbf{a}$ and $\mathbf{b}$, in $[0, \pi]$.

Denote

$$t = \cos\langle \mathbf{p}, \mathbf{r} \rangle + \cos\langle \mathbf{r}, \mathbf{q} \rangle + \cos\langle \mathbf{q}, \mathbf{p} \rangle + 1,$$

and hence

$$\begin{cases} \mathbf{u} \cdot \mathbf{v} = ij|\mathbf{r}|^2|\mathbf{p}||\mathbf{q}|t, \\ \mathbf{u} \cdot \mathbf{w} = ik|\mathbf{r}||\mathbf{p}||\mathbf{q}|^2 t, \\ \mathbf{v} \cdot \mathbf{w} = jk|\mathbf{r}||\mathbf{p}|^2|\mathbf{q}|t. \end{cases}$$

Since $i, j, k > 0$, and $|\mathbf{p}|, |\mathbf{q}|, |\mathbf{r}| > 0$ since none of P, Q, R are at O, we must have

$$\operatorname{sgn}(\mathbf{u} \cdot \mathbf{v}) = \operatorname{sgn}(\mathbf{u} \cdot \mathbf{w}) = \operatorname{sgn}(\mathbf{v} \cdot \mathbf{w}) = \operatorname{sgn} t,$$

where $\operatorname{sgn} : \mathbb{R} \to \{-1, 0, -1\}$ is the sign function defined as

$$\operatorname{sgn} x = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

But the sign of a dot product also corresponds to the angle between two non-collinear non-zero vectors, since this resembles the sign of the cosine of the angle between them:

$$\begin{aligned} \operatorname{sgn} \mathbf{a} \cdot \mathbf{b} &= \operatorname{sgn} |\mathbf{a}||\mathbf{b}| \cos\langle \mathbf{a}, \mathbf{b} \rangle \\ &= \operatorname{sgn} \cos\langle \mathbf{a}, \mathbf{b} \rangle \\ &= \begin{cases} 1, & \langle a, b \rangle \text{ is acute}, \\ 0, & \langle a, b \rangle \text{ is right-angle}, \\ -1, & \langle a, b \rangle \text{ is obtuse}. \end{cases} \end{aligned}$$

This means the angles between $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{w}$, $\mathbf{v}$ and $\mathbf{w}$ must all be acute, obtuse, or right angles. This is exactly what is desired, and finishes our proof.