2024.2 Question 3

1. The line NP has gradient
$$m_{NP} = \frac{\sin\theta - 0}{\cos\theta - (-1)} = \frac{\sin\theta}{\cos\theta + 1},$$
and hence it has equation
$$l_{NP} : y = \frac{\sin\theta}{\cos\theta + 1} \cdot (x + 1).$$
When $x = 0$, we have
$$\begin{aligned} q &= \frac{\sin\theta}{\cos\theta + 1} \\ &= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} - 1 + 1} \\ &= \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \\ &= \tan\frac{\theta}{2}. \end{aligned}$$

2. (a)
$$\begin{aligned} \text{RHS} &= \tan\frac{1}{2}\left(\theta + \frac{1}{2}\pi\right) \\ &= \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) \\ &= \frac{\tan\frac{\theta}{2} + \tan\frac{\pi}{4}}{1 - \tan\frac{\theta}{2}\tan\frac{\pi}{4}} \\ &= \frac{q+1}{1-q} \\ &= f_1(q), \end{aligned}$$
as desired.

(b) Let the coordinates of P_1 be $(\cos\varphi, \sin\varphi)$, and hence we must have
$$\begin{aligned} f_1(q) &= \tan\frac{1}{2}\varphi \\ \tan\frac{1}{2}\left(\theta + \frac{1}{2}\pi\right) &= \tan\frac{1}{2}\varphi \\ \varphi &= \theta + \frac{1}{2}\pi, \end{aligned}$$
and so P_1 is the image of P being rotated through an angle of π counterclockwise about the origin.

3. (a) The coordinates of P_2 are $\left(\cos\left(\theta + \frac{1}{3}\pi\right), \sin\left(\theta + \frac{1}{3}\pi\right)\right)$, and hence we must have that
$$\begin{aligned} f_3(q) &= \tan\frac{1}{2}\left(\theta + \frac{1}{3}\pi\right) \\ &= \tan\left(\frac{\theta}{2} + \frac{\pi}{6}\right) \\ &= \frac{\tan\frac{\theta}{2} + \tan\frac{\pi}{6}}{1 - \tan\frac{\theta}{2}\tan\frac{\pi}{6}} \\ &= \frac{q + \frac{1}{\sqrt{3}}}{1 - q \cdot \frac{1}{\sqrt{3}}} \\ &= \frac{1 + \sqrt{3}q}{\sqrt{3} - q}. \end{aligned}$$

(b) Notice that $f_3(q) = f_1(-q) = \tan \frac{1}{2}\left(-\theta + \frac{1}{2}\pi\right)$, and so the coordinates of P_3 must be

$$\left(\cos\left(\frac{1}{2}\pi - \theta\right), \sin\left(\frac{1}{2}\pi - \theta\right)\right),$$

which is $P_3(\sin\theta, \cos\theta)$, a reflection of P in the line $y = x$.

(c) P_4 must be the image of P under the following transformations:

- Rotation counterclockwise by $\frac{1}{3}\pi$ about the origin O;
- Reflection in the line $y = x$;
- Rotation clockwise by $\frac{1}{3}\pi$ about the origin O.

This is precisely the reflection in which the axis after the second step is $y = x$. Hence, the axis of this reflection has an angle of $\frac{1}{4}\pi - \frac{1}{3}\pi = \frac{1}{12}\pi$ with the positive x-axis.

P_4 is the image of P reflected in the line which makes an angle of $-\frac{\pi}{12}$ with the positive x-axis, passing through the origin.