2024.2 Question 2

1. By Newton's binomial theorem, we have

$$
\begin{aligned}
(8+x^3)^{-1} &= \frac{1}{8}\left(1+\left(\frac{x}{2}\right)^3\right)^{-1} \\
&= \frac{1}{8}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^{3k},
\end{aligned}
$$

and this is valid for

$$\left|\frac{x}{2}\right| < 1, |x| < 2,$$

as desired.

Hence,

$$
\begin{aligned}
\int_0^1 \frac{x^m}{8+x^3}\,\mathrm{d}x &= \int_0^1 \frac{1}{8}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^{3k} x^m\,\mathrm{d}x \\
&= \frac{1}{8}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3k}}\int_0^1 x^{3k+m}\,\mathrm{d}x \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\left[\frac{x^{3k+m+1}}{3k+m+1}\right]_0^1 \\
&= \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2^{3(k+1)}}\cdot\frac{1}{3k+m+1}\right),
\end{aligned}
$$

as desired.

2. Let $m=2$, and we have

$$\int_0^1 \frac{x^2}{8+x^3}\,\mathrm{d}x = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2^{3(k+1)}}\cdot\frac{1}{3k+3}\right).$$

Let $m=1$, and we have

$$\int_0^1 \frac{x}{8+x^3}\,\mathrm{d}x = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2^{3(k+1)}}\cdot\frac{1}{3k+2}\right).$$

Let $m=0$, and we have

$$\int_0^1 \frac{x}{8+x^3}\,\mathrm{d}x = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2^{3(k+1)}}\cdot\frac{1}{3k+1}\right).$$

Hence,

$$
\begin{aligned}
\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\left(\frac{1}{3k+3}-\frac{2}{3k+2}+\frac{4}{3k+1}\right) &= \int_0^1 \frac{x^2}{8+x^3}\,\mathrm{d}x - 2\int_0^1 \frac{x}{8+x^3}\,\mathrm{d}x + 4\int_0^1 \frac{\mathrm{d}x}{8+x^3} \\
&= \int_0^1 \frac{x^2-2x+4}{8+x^3}\,\mathrm{d}x \\
&= \int_0^1 \frac{x^2-2x+4}{(x+2)(x^2-2x+4)}\,\mathrm{d}x \\
&= \int_0^1 \frac{\mathrm{d}x}{x+2} \\
&= [\ln|x+2|]_0^1 \\
&= \ln 3 - \ln 2.
\end{aligned}
$$

3. Using partial fractions, let A' and B' be real constants such that

$$\begin{aligned}\frac{72(2k+1)}{(3k+1)(3k+2)} &= \frac{A'}{3k+1} + \frac{B'}{3k+2} \\ &= \frac{3(A'+B')k + (2A'+B')}{(3k+1)(3k+2)}.\end{aligned}$$

Hence, we have

$$\begin{cases} 3(A'+B') = 72 \cdot 2 = 144, \\ 2A' + B' = 72. \end{cases}$$

Therefore, $(A', B') = (24, 24)$.

Let

$$A = \int_0^1 \frac{\mathrm{d}x}{8+x^3}, B = \int_0^1 \frac{x \, \mathrm{d}x}{8+x^3}, C = \int_0^1 \frac{x^2 \, \mathrm{d}x}{8+x^3},$$

and what is desired is $24(A+B)$.

From the previous part, we can see that $4A - 2B + C = \ln 3 - \ln 2$.

Also,

$$\begin{aligned}2A + B &= \int_0^1 \frac{(2+x)\, \mathrm{d}x}{8+x^3} \\ &= \int_0^1 \frac{\mathrm{d}x}{x^2-2x+4} \\ &= \int_0^1 \frac{\mathrm{d}x}{(x-1)^2+3} \\ &= \frac{1}{\sqrt{3}} \left[\arctan\left(\frac{x-1}{\sqrt{3}}\right)\right]_0^1 \\ &= \frac{1}{\sqrt{3}} \cdot \left[\arctan 0 - \arctan\left(-\frac{1}{\sqrt{3}}\right)\right] \\ &= \frac{1}{\sqrt{3}} \cdot \frac{\pi}{6} \\ &= \frac{\pi}{6\sqrt{3}}.\end{aligned}$$

We also have

$$\begin{aligned}C &= \int_0^1 \frac{x^2 \, \mathrm{d}x}{8+x^3} \\ &= \frac{1}{3}\left[\ln(8+x^3)\right]_0^1 \\ &= \frac{1}{3}\left[\ln 9 - \ln 8\right] \\ &= \frac{2}{3} \ln 3 - \ln 2.\end{aligned}$$

Hence, we have

$$4A - 2B = \ln 3 - \ln 2 - \frac{2}{3}\ln 3 + \ln 2 = \frac{1}{3} \ln 3,$$

and hence $2A - B = \frac{1}{6} \ln 3$.

Therefore,

$$4A = \frac{1}{6} \ln 3 + \frac{\pi}{6\sqrt{3}},$$

and hence

$$A = \frac{\ln 3}{24} + \frac{\pi}{24\sqrt{3}}.$$

Subtracting two of this from $2A + B$ gives

$$B = \frac{\pi}{6\sqrt{3}} - \frac{\ln 3}{12} - \frac{\pi}{12\sqrt{3}} = \frac{\pi}{12\sqrt{3}} - \frac{\ln 3}{12},$$

and hence what is desired is

$$\begin{aligned} 24(A+B) &= 24\left(\frac{\pi}{24\sqrt{3}} + \frac{\pi}{12\sqrt{3}} + \frac{\ln 3}{24} - \frac{\ln 3}{12}\right) \\ &= 24\left(\frac{\pi}{8\sqrt{3}} - \frac{\ln 3}{24}\right) \\ &= \pi\sqrt{3} - \ln 3, \end{aligned}$$

which gives $a = 3, b = 3$.