2024.2 Question 12

1. Let X_i be the number that the ith player receives, and let Ada be the first player. We have

$$\begin{aligned}
\mathrm{P}(X_1 = a, X_2 > X_1, X_3 > X_1, \cdots, X_k > X_1) &= \mathrm{P}(X_1 = a, X_2 > a, X_3 > a, \cdots, X_k > a) \\
&= \mathrm{P}(X_1 = a)\,\mathrm{P}(X_2 > a)\,\mathrm{P}(X_3 > a) \cdots \mathrm{P}(X_k > a) \\
&= \frac{1}{n} \cdot \frac{n-a}{n} \cdot \frac{n-a}{n} \cdots \frac{n-a}{n} \\
&= \frac{(n-a)^{k-1}}{n^k}.
\end{aligned}$$

Hence, the probability of Ada winning this is

$$\begin{aligned}
\mathrm{P}(X_2 > X_1, X_3 > X_1, \cdots, X_k > X_1) &= \sum_{a=1}^{n-1} \mathrm{P}(X_1 = a, X_2 > X_1, X_3 > X_1, \cdots, X_k > X_1) \\
&= \sum_{a=1}^{n-1} \frac{(n-a)^{k-1}}{n^k} \\
&= \frac{1}{n^k} \sum_{a=1}^{n-1} a^{k-1},
\end{aligned}$$

and the probability of there being a winner is the sum of the probabilities of each player winning, which are all equal to the probability of Ada winning by symmetry, and hence is equal to

$$k \cdot \frac{1}{n^k} \sum_{a=1}^{n-1} a^{k-1} = \frac{k}{n^k} \sum_{a=1}^{n-1} a^{k-1}.$$

If $k = 4$, then this probability is given by

$$\begin{aligned}
\mathrm{P} &= \frac{4}{n^4} \sum_{a=1}^{n-1} a^3 \\
&= \frac{4}{n^4} \cdot \frac{(n-1)^2 n^2}{4} \\
&= \frac{(n-1)^2}{n^2},
\end{aligned}$$

precisely as desired.

2. Similarly, let X_i be the number that the ith player receives, and let Ada be the first player, and Bob be the second player. We have

$$\begin{aligned}
&\mathrm{P}(X_1 = a, X_2 = a + d + 1, X_1 < X_3 < X_2, \cdots, X_1 < X_k < X_2) \\
&= \mathrm{P}(X_1 = a, X_2 = a + d + 1, a < X_3 < a + d + 1, \cdots, a < X_k < a + d + 1) \\
&= \mathrm{P}(X_1 = a)\,\mathrm{P}(X_2 = a + d + 1)\,\mathrm{P}(a < X_3 < a + d + 1) \cdots \mathrm{P}(a < X_k < a + d + 1) \\
&= \frac{1}{n} \cdot \frac{1}{n} \cdot \frac{d}{n} \cdots \frac{d}{n} \\
&= \frac{d^{k-2}}{n^k}.
\end{aligned}$$

Hence, the probability that both Ada and Bob winning this is

$$
\begin{aligned}
& \mathrm{P}(X_1 < X_3 < X_2, \cdots, X_1 < X_k < X_2) \\
&= \sum_{d=1}^{n-2} \sum_{a=1}^{n-d-1} \mathrm{P}(X_1 = a, X_2 = a + d + 1, X_1 < X_3 < X_2, \cdots, X_1 < X_k < X_2) \\
&= \sum_{d=1}^{n-2} \sum_{a=1}^{n-d-1} \frac{d^{k-2}}{n^k} \\
&= \sum_{d=1}^{n-2} \frac{(n-d-1)d^{k-2}}{n^k} \\
&= \frac{1}{n^k} \sum_{d=1}^{n-2} (n-d-1)d^{k-2} \\
&= \frac{1}{n^k} \left[(n-1) \sum_{d=1}^{n-2} d^{k-2} - \sum_{d=1}^{n-2} d^{k-1} \right].
\end{aligned}
$$

Hence, the probability that there are two winners in this game is the sum of the probabilities of each ordered pair of players winning (since there is one winning by having a bigger number, and one winning by having a smaller number), and hence is equal to

$$
2 \cdot \binom{k}{2} \cdot \frac{1}{n^k} \left[(n-1) \sum_{d=1}^{n-2} d^{k-2} - \sum_{d=1}^{n-2} d^{k-1} \right].
$$

When $k = 4$, the probability is

$$
\begin{aligned}
\mathrm{P} &= 2 \cdot \binom{4}{2} \cdot \frac{1}{n^4} \left[(n-1) \sum_{d=1}^{n-2} d^2 - \sum_{d=1}^{n-2} d^3 \right] \\
&= 2\dot{6} \cdot \frac{1}{n^4} \left[\frac{(n-1)(n-2)(n-1)(2n-3)}{6} - \frac{(n-2)^2(n-1)^2}{4} \right] \\
&= 12 \cdot \frac{1}{n^4} \cdot (n-1)^2(n-2) \left[\frac{2(2n-3) - 3(n-2)}{12} \right] \\
&= \frac{(n-1)^2(n-2)}{n^4} \cdot n \\
&= \frac{(n-2)(n-1)^2}{n^3}.
\end{aligned}
$$

The probability of there being a winner due to having the biggest number (denote this event as B), is the same as there being a winner due to having the lowest number (denote this event as L), which are both equal to the answer to the first part of the question:

$$
\mathrm{P}(B) = \mathrm{P}(L) = \frac{(n-1)^2}{n^2}.
$$

The event of having two winners is B, L and the event of having precisely one winner is $B, \bar{L}$ or $L, \bar{B}$. By the inclusion-exclusion principle, the probability of having precisely one winner is given by

$$
\begin{aligned}
\mathrm{P} &= \mathrm{P}(B) + \mathrm{P}(L) - 2\,\mathrm{P}(B, L) \\
&= 2 \cdot \frac{(n-1)^2}{n^2} - 2 \cdot \frac{(n-2)(n-1)^2}{n^3} \\
&= \frac{2(n-1)^2}{n^3} \cdot [n - (n-2)] \\
&= \frac{4(n-1)^2}{n^3}.
\end{aligned}
$$

This probability is smaller than $\mathrm{P}(B, L)$, if and only if

$$\begin{aligned}\frac{4(n-1)^2}{n^3} &< \frac{(n-2)(n-1)^2}{n^3} \\ 4 &< n-2 \\ n &> 6,\end{aligned}$$

and hence the minimum value of n for this is 7.