2024.2 Question 1

1. In the $n+k$ integers, the first one is c, and the final one is $c+n+k-1$.

   In the n integers, the first one is $c+n+k$, and the final one is $c+2n+k-1$.

   Hence, the sums are equal if and only if

$$\begin{aligned}\frac{(n+k)[c+(c+n+k-1)]}{2} &= \frac{n[(c+n+k)+(c+2n+k-1)]}{2}\\(n+k)(2c+n+k-1) &= n(2c+3n+2k-1)\\2cn+n^2+nk-n+2ck+kn+k^2-k &= 2cn+3n^2+2kn-1\\2ck+k^2 &= 2n^2+k,\end{aligned}$$

   as desired. All the above steps are reversible.

2. (a) When $k=1$, $2c+1=2n^2+1$, and $c=n^2$.
   Hence,
$$(c,n)\in\left\{(t^2,t)\mid t\in\mathbb{N}\right\},$$
   and n can take all positive integers.

   (b) When $k=2$, $4c+4=2n^2+2$, and $2c=n^2-1$.
   By parity, n must be odd. Let $n=2t-1$ for $t\in\mathbb{N}$, and we have
$$2c=(2t-1)^2-1=4t^2-4t,$$
   and hence
$$c=2t^2-2t.$$
   Hence,
$$(c,n)\in\left\{(2t^2-2t,2t-1)\mid t\in\mathbb{N}\right\},$$
   and n can take all odd positive integers.

3. If $k=4$, we have $8c+16=2n^2+4$, and hence $n^2=4c+6$.

   By considering modulo 4, the only quadratic residues modulo 4 are 0 and 1, but the right-hand side equation is congruent to 2 modulo 4.

   Hence, there are no solutions for n and c.

4. When $c=1$, we have $2n^2+k=2k+k^2$, and hence $2n^2=k^2+k$.

   (a) When $k=1$, $k^2+k=2$, and so $(n,k)=(1,1)$ satisfies the equation.
   When $k=8$, $k^2+k=64+8=72$, and so $(n,k)=(6,8)$ satisfies the equation.

   (b) Given that $2N^2=K^2+K$, notice that

$$\begin{aligned}(2N'^2)-(K'^2+K') &= 2(3N+2K+1)^2-(4N+3K+1)^2-(4N+3K+1)\\&= 2(9N^2+4K^2+1+12NK+6N+4K)\\&\quad-(16N^2+9K^2+1+24NK+8N+6K)\\&\quad-(4N+3K+1)\\&= 2N^2-K^2-K\\&= 2N^2-(K^2+K)\\&= 2N^2-2N^2\\&= 0,\end{aligned}$$

   and this means that
$$2N'^2=K'^2+K',$$
   and hence
$$(N',K')=(3N+2K+1,4N+3K+1)$$
   is another pair of solution for (n,k).

   (c) When $(n,k)=(6,8)$, $3n+2k+1=35$, $4n+3k+1=49$, and $(n,k)=(35,49)$ is also possible.
   When $(n,k)=(35,49)$, $3n+2k+1=204, 4n+3k+1=288$, and $(n,k)=(204,288)$ is also possible.