2023.3 Question 8

1. By differentiating, we have
$$f'(x) = e^{-x} - xe^{-x} = e^{-x} - f(x),$$
and
$$f''(x) = -e^{-x} - f'(x).$$
Hence,
$$\begin{aligned}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y &= f''(x) + 2f'(x) + f(x)\\ &= -e^{-x} - f'(x) + f'(x) + e^{-x} - f(x) + f(x)\\ &= 0\end{aligned}$$
as desired.
Evaluating y and y' at $x = 0$ gives us
$$y|_{x=0} = f(0) = 0 \cdot e^{-0} = 0$$
and
$$y'|_{x=0} = f'(0) = e^{-0} - f(0) = 1 - 0 = 1.$$
For the final part, we factorise $f'(x)$ to get $f'(x) = (1 - x)e^{-x}$.
$e^{-x} > 0$ for all x. Therefore, for $x \leq 1$, $1 - x \geq 0$, and hence $f'(x) \geq 0$.

2. We let $g_1(x) = f(x) = xe^{-x}$, and we can immediately see that this differential equation is satisfied by $x \leq 1$.
For $y = g_2(x)$ where $x \geq 1$, we notice $g_2(1) = g_1(1) = 1 \cdot e^{-1} = \frac{1}{e}$, and $g_2'(1) = g_1'(1) = f'(1) = e^{-1} - f(1) = \frac{1}{e} - \frac{1}{e} = 0$.
If $g_2'(x) \geq 0$ for $x \geq 1$, then g_2 and g_1 satisfies the same differential equation and boundary conditions (at $x = 1$), which means they are the same solution.
However, this is impossible since $g_1'(x) < 0$ for $x > 1$.
Therefore, it must be the case that $g_2'(x) \leq 0$ for $x \geq 1$, and hence we have $g_2''(x) - 2g_2'(x) + g_2(x) = 0$ as our differential equation.
The characteristic equation solves to $\lambda_{1,2} = 1$, and hence the general solution to g_2 is $g_2(x) = (A + Bx)e^x$.
By differentiating, we have
$$g_2'(x) = Be^x + (A + Bx)e^x = Be^x + g_2(x).$$
Considering the boundary conditions, we first have $g_2(1) = \frac{1}{e}$, meaning that $(A + B)e = \frac{1}{e}$, and hence $A + B = e^{-2}$.
We have as well $g_2'(1) = 0$, and hence $0 = B \cdot e + \frac{1}{e}$, giving us $B = -e^{-2}$.
Therefore, $A = 2e^{-2}$, and hence
$$\begin{aligned}g_2(x) &= \left(2e^{-2} - e^{-2}x\right) e^x\\ &= e^{-2}(2 - x)e^x\\ &= (2 - x)e^{x-2}.\end{aligned}$$

3. We notice that $g_2(x) = g_1(2 - x)$, and hence $g_2(1 + x) = g_1(1 - x)$. This means they are symmetric about the line $x = 1$.

4. We first consider the range that x is in. We replace x with $c-x$ to acquire

$$\begin{aligned} r \leq c-x \leq s &\iff -r \geq -c+x \geq -s \\ &\iff -r+c \geq x \geq -s+c \\ &\iff -s+c \leq x \leq -r+c. \end{aligned}$$

In other words,

$$x \in [-s+c, -r+c] \iff c-x \in [r, s].$$

If $y = k(c-x)$, then we have $y' = (-1) \cdot k'(c-x)$, and $y'' = (-1)^2 \cdot k''(c-x) = k''(c-x)$. Therefore,

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - p\frac{\mathrm{d}y}{\mathrm{d}x} + qy &= k''(c-x) + pk'(c-x) + qk(c-x) \\ &= k''(t) + pk'(t) + qk(t) \end{aligned}$$

for $t = c-x \in [r, s]$.

Since $y = k(x)$ is a solution to the original differential equation for $r \leq x \leq s$, we must have $k''(t) + pk'(t) + qk(t) = 0$, and therefore $y = k(c-x)$ satisfies the new differential equation for $-s+c \leq x \leq -r+c$.

5. By differentiating h, we have

$$h'(x) = -e^{-x}\sin x + e^{-x}\cos x = e^{-x}(\cos x - \sin x).$$

Therefore,

$$\begin{aligned} h'\left(\frac{1}{4}\pi\right) &= e^{-\frac{1}{4}\pi}\left(\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) \\ &= e^{-\frac{1}{4}\pi}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) \\ &= 0. \end{aligned}$$

Similarly,

$$h'\left(-\frac{3}{4}\pi\right) = e^{\frac{3}{4}\pi}\left(-\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right)\right) = 0.$$

For $x \in \left[-\frac{3}{4}\pi, \frac{1}{4}\pi\right]$, the differential equation satisfied by h without the absolute value sign is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$$

since $h'(x) \geq 0$.

(a) Let $c = \frac{\pi}{2}$. For $x \in \left[\frac{\pi}{2} - \frac{\pi}{4}, \frac{\pi}{2} + \frac{3\pi}{4}\right] = \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$, by the previous lemma, $y = h\left(\frac{\pi}{2} - x\right)$ must be a solution to

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0.$$

Notice that

$$y' = -h'\left(\frac{\pi}{2} - x\right),$$

and that $x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right] \iff \frac{\pi}{2} - x \in \left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]$, and hence $h'\left(\frac{\pi}{2} - x\right) \geq 0$, which means $y' \leq 0$. Therefore, in $x \in \left[\frac{1}{4}\pi, \frac{5}{4}\pi\right]$, $y = h\left(\frac{\pi}{2} - x\right)$ satisfies

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| + 2y = 0,$$

which is the original differential equation.

We show next that this is continuously differentiable at $x = \frac{1}{4}\pi$.
It is continuous since
$$h\left(\frac{1}{4}\pi\right) = h\left(\frac{\pi}{2} - \frac{1}{4}\pi\right) = h\left(\frac{1}{4}\pi\right).$$
We have $h'(x)|_{x=\frac{1}{4}\pi} = 0$, and
$$-h'\left(\frac{\pi}{2} - x\right)\Big|_{x=\frac{1}{4}\pi} = -h'\left(\frac{\pi}{4}\right) = 0,$$
so it is continuously differentiable at $\frac{1}{4}\pi$.
Hence,
$$\begin{aligned} y &= h\left(\frac{\pi}{2} - x\right) \\ &= e^{x-\frac{\pi}{2}} \sin\left(\frac{\pi}{2} - x\right) \\ &= e^{x-\frac{\pi}{2}} \cos x, \end{aligned}$$
for $x \in \left[\frac{1}{4}\pi, \frac{5}{4}\pi\right]$.

(b) As shown above, for $x \in \left[\frac{1}{4}\pi, \frac{5}{4}\pi\right]$, $y = h\left(\frac{\pi}{2} - x\right)$ satisfies
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0.$$
Let $c = \frac{5\pi}{2}$. For $x \in \left[\frac{5\pi}{2} - \frac{5}{4}\pi, \frac{5\pi}{2} - \frac{1}{4}\pi\right] = \left[\frac{5}{4}\pi, \frac{9}{4}\pi\right]$,
$$y = h\left(\frac{\pi}{2} - \left(\frac{5\pi}{2} - x\right)\right) = h(x - 2\pi)$$
satisfies
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0.$$
We have
$$y' = h'(x - 2\pi) = h'\left(\frac{\pi}{2} - \left(\frac{5\pi}{2} - x\right)\right),$$
and $x \in \left[\frac{5}{4}\pi, \frac{9}{4}\pi\right] \iff \frac{5}{2}\pi - x \in \left[\frac{1}{4}\pi, \frac{5}{4}\pi\right]$, and this therefore means $h'\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right) \geq 0$.
Hence, in $x \in \left[\frac{5}{4}\pi, \frac{9}{4}\pi\right]$, $y = h(x - 2\pi)$ satisfies
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| + 2y = 0.$$
We show next that this is continuously differentiable at $x = \frac{5}{4}\pi$.
It is continuous since
$$h\left(\frac{\pi}{2} - \frac{5}{4}\pi\right) = h\left(-\frac{3}{4}\pi\right) = h\left(\frac{5}{4}\pi - 2\pi\right).$$
We have
$$h'\left(\frac{\pi}{2} - x\right)\Big|_{x=\frac{5}{4}\pi} = -\left.h'(x)\right|_{x=-\frac{3}{4}\pi} = -0 = 0,$$
and
$$\left.h'(x - 2\pi)\right|_{x=\frac{5}{4}\pi} = \left.h'(x)\right|_{x=-\frac{3}{4}\pi} = 0,$$
and so it is continuously differentiable at $x = \frac{5}{4}\pi$.
Therefore,
$$\begin{aligned} y &= h(x - 2\pi) \\ &= e^{-x+2\pi} \sin(x - 2\pi) \\ &= e^{2\pi-x} \sin x \end{aligned}$$
for $x \in \left[\frac{5}{4}\pi, \frac{9}{4}\pi\right]$.