## 2023.3 Question 5

1. If $x, y$ are both non-zero,

$$\begin{aligned}\frac{1}{x}+\frac{2}{y}&=\frac{2}{7}\\ 7y+2\cdot 7x&=2xy\\ 2xy-14x-7y&=0\\ 2xy-14x-7y+49&=49\\ 2x(y-7)-7(y-7)&=49\\ (2x-7)(y-7)&=49.\end{aligned}$$

We must have $2x-7\geq 2\cdot 1-7=-5$ and $y-7\geq 1-7=-6$.

$2x-7$ and $y-7$ are both integers, and we do casework considering expressing 49 into a product of two integers that are both not less than $-6$.

- $49=1\times 49$, $2x-7=1$ and $y-7=49$, giving us $(x,y)=(4,56)$.
- $49=7\times 7$, $2x-7=7$ and $y=7=7$, giving us $(x,y)=(7,14)$.
- $49=49\times 1$, $2x-7=49$ and $x-7=1$, giving us $(x,y)=(28,8)$.

Since all $x, y$ are non-zero, we can conclude that the solutions are $(x,y)=(4,56),(7,14),(28,8)$.

2. We have

$$\begin{aligned}p^2+pq+q^2&=n^2\\ p^2+2pq+q^2&=n^2+pq\\ (p+q)^2&=n^2+pq\\ (p+q)^2-n^2&=pq\\ (p+q+n)(p+q-n)&=pq.\end{aligned}$$

We must have $p+q+n>p+q-n$ since $n$ is a positive integer. We have $p+q+n>p,q>1>0$. It must be the case that $p+q-n$ is positive as well.

Therefore, $p+q+n$ cannot be $1,p,q$, and it must be the case that $p+q+n=pq$ and $p+q-n=1$.

Therefore, $p+q=n+1$, and $pq=p+q+n=2n+1$.

Hence, $p, q$ are solutions to the quadratic equation in $t$

$$t^2-(n+1)t+(2n+1)=0.$$

Solving this gives us

$$\begin{aligned}p,q&=\frac{(n+1)\pm\sqrt{(n+1)^2-4\cdot(2n+1)}}{2}\\ &=\frac{(n+1)\pm\sqrt{n^2-6n-3}}{2}.\end{aligned}$$

We have $n^2-6n-3=(n-3)^2-12$ must be a perfect square for $p,q$ to be rational (and they are since all integers are rational).

Consider $a,b\geq 0$, $a,b\in\mathbb{N}$ such that $a^2-b^2=(a+b)(a-b)=12$.

$a+b$ and $a-b$ must take the same odd-even parity, and the only possibility is therefore $a+b=6$ and $a-b=2$, solving to $(a,b)=(4,2)$.

Therefore, $n-3=4$, $n=7$, and we solve for

$$p,q=\frac{8\pm\sqrt{49-42-3}}{2}=4\pm 1$$

and $(p,q)=(3,5),(5,3)$ are indeed primes, and $n=7$.

3. If $p+q-n \geq p$, then $q \geq n$, and for the original equation,

$$\text{LHS} = p^3 + q^3 + 3pq^2 > q^3 \geq n^3 = \text{RHS},$$

and hence LHS > RHS is impossible. Hence, $p+q-n < p$.

It must also be the case for $p+q-n < q$.

We have

$$\begin{aligned} p^3 + q^3 + 3pq^2 &= n^3 \\ p^3 + q^3 + 3pq^2 + 3p^2q &= n^3 + 3p^2q \\ (p+q)^3 &= n^3 + 3p^2q \\ (p+q)^3 - n^3 &= 3p^2q \\ (p+q-n)\left[(p+q)^2 + (p+q)\cdot n + n^2\right] &= 3p^2q. \end{aligned}$$

The factors of $3p^2q$ are (given $p$ and $q$ are prime),

$$1, 3, p, q, 3p, 3q, p^2, pq, 3p^2, 3pq, p^2q, 3p^2q,$$

and since $p+q-n < p$ and $p+q-n < q$, it must be either the case that $p+q-n = 1$ or $p+q-n = 3$.

- If $p+q-n = 1$, then $p+q = n+1$, we have

$$\begin{aligned} (p+q)^2 + (p+q)n + n^2 &= 3p^2q \\ (n+1)^2 + (n+1)n + n^2 &= 3p^2q \\ 3n^2 + 3n + 2 &= 3p^2q. \end{aligned}$$

  The left-hand side is congruent to 1 modulo 3, while the right-hand side is a multiple of 3, so this is impossible.

- If $p+q-n = 3$, $p+q = n+3$, we have

$$\begin{aligned} (p+q)^2 + (p+q)n + n^2 &= p^2q \\ (n+3)^2 + (n+3)n + n^2 &= p^2q \\ 3n^2 + 9n + 9 &= p^2q \\ 3(n^2 + 3n + 3) &= p^2q. \end{aligned}$$

  Therefore, $3 \mid p^2q$, and hence $3 \mid p$ or $3 \mid q$, and hence either $p$ or $q$ must be 3 and the other one is $n$. However, we have concluded that $p+q-n < p \iff q < n$ and $p+q-n < q \iff p < n$, which makes this impossible.

This shows that it is impossible for primes $p, q$ and integer $n$ such that $p^3 + q^3 + 3pq^2 = n^3$, which shows that there are no primes $p, q$ such that $p^3 + q^3 + 3pq^2$ is the cube of an integer.