## 2023.3 Question 2

1. If the two curves meet at $\theta = \alpha$, then $\alpha$ must satisfy that

$$k(1 + \sin\alpha) = k + \cos\alpha.$$

Subtracting $k$ on both sides, we have

$$k\sin\alpha = \cos\alpha,$$

and since $k > 1 > 0$, $\sin\alpha$ and $\cos\alpha$ cannot be simultaneously zero, they must both be non-zero. Dividing through both sides by $\cos\alpha$ gives

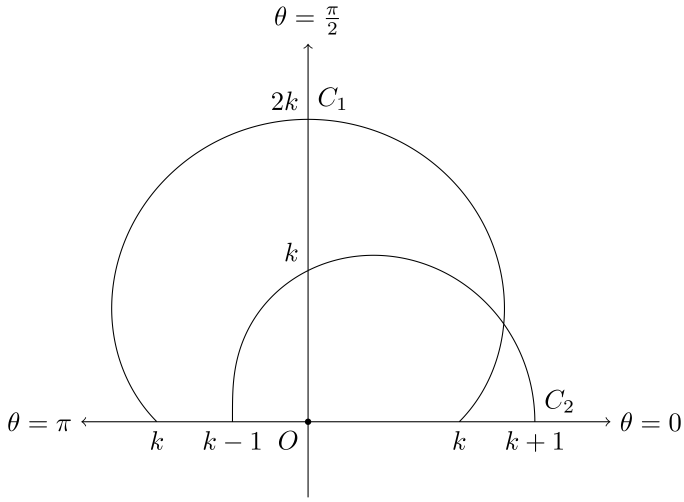
$$k\tan\alpha = 1$$

and hence

$$\tan\alpha = \frac{1}{k}$$

as desired.

The curves are as follows.




2. The area of $A$ is given by

$$\begin{aligned}
[A] &= \frac{1}{2} \cdot \int_0^\alpha (k(1+\sin\theta))^2 \,\mathrm{d}\theta \\
&= \frac{k^2}{2} \cdot \int_0^\alpha \left(1 + 2\sin\theta + \sin^2\theta\right) \mathrm{d}\theta \\
&= \frac{k^2}{2} \cdot \int_0^\alpha \left(1 + 2\sin\theta + \frac{1-\cos 2\theta}{2}\right) \mathrm{d}\theta \\
&= \frac{k^2}{2} \cdot \left[\frac{3}{2} \cdot \theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^\alpha \\
&= \frac{k^2}{2} \cdot \left[\left(\frac{3}{2}\alpha - 2\cos\alpha - \frac{1}{4}\sin 2\alpha\right) - \left(0 - 2 - \frac{1}{4} \cdot 0\right)\right] \\
&= \frac{k^2}{2}\left[\frac{3}{2}\alpha - 2\cos\alpha - \frac{1}{2}\sin\alpha\cos\alpha + 2\right] \\
&= \frac{k^2}{4}\left(3\alpha - \sin\alpha\cos\alpha\right) + k^2\left(1 - \cos\alpha\right).
\end{aligned}$$

3. The area of $B$ is given by

$$\begin{aligned}
[B] &= \frac{1}{2} \cdot \int_{\alpha}^{\pi} (k + \cos\theta)^2 \, \mathrm{d}\theta \\
&= \frac{1}{2} \cdot \int_{\alpha}^{\pi} \left(k^2 + 2k\cos\theta + \cos^2\theta\right) \mathrm{d}\theta \\
&= \frac{1}{2} \cdot \int_{\alpha}^{\pi} \left(k^2 + 2k\cos\theta + \frac{1+\cos 2\theta}{2}\right) \mathrm{d}\theta \\
&= \frac{1}{2} \cdot \left[\left(k^2 + \frac{1}{2}\right)\theta + 2k\sin\theta + \frac{\sin 2\theta}{4}\right]_{\alpha}^{\pi} \\
&= \frac{1}{2} \cdot \left[\left(\left(k^2 + \frac{1}{2}\right)\pi + 2k \cdot 0 + \frac{0}{4}\right) - \left(\left(k^2 + \frac{1}{2}\right)\alpha + 2k \cdot \sin\alpha + \frac{\sin 2\alpha}{4}\right)\right] \\
&= \frac{1}{2} \cdot \left[\left(k^2 + \frac{1}{2}\right)(\pi - \alpha) - 2k\sin\alpha - \frac{\sin\alpha\cos\alpha}{2}\right] \\
&= \frac{1}{4} \cdot \left(2k^2\pi + \pi - 2k^2\alpha - \alpha - 4k\sin\alpha - \sin\alpha\cos\alpha\right).
\end{aligned}$$

4. $T$ is given by

$$\begin{aligned}
T &= \frac{1}{2} \cdot \int_{0}^{\pi} (k + \cos\theta)^2 \, \mathrm{d}\theta \\
&= \frac{1}{2} \cdot \left[\left(k^2 + \frac{1}{2}\right)\theta + 2k\sin\theta + \frac{\sin 2\theta}{4}\right]_{0}^{\pi} \\
&= \frac{1}{4} \cdot \left(2k^2\pi + \pi - 2k^2 \cdot 0 - 0 - 4k \cdot 0 - 0 \cdot 1\right) \\
&= \frac{\pi\left(2k^2 + 1\right)}{4}.
\end{aligned}$$

As $k \to \infty$, $\frac{1}{k} = \tan\alpha \to 0^+$, and therefore,

$$\alpha, \sin\alpha, \tan\alpha \approx \frac{1}{k}$$

and

$$\cos\alpha \approx 1 - \frac{1}{2k^2}.$$

Therefore, considering only terms with the highest power of $k$

$$\begin{aligned}
[A] &= \frac{k^2}{4}(3\alpha - \sin\alpha\cos\alpha) + k^2(1 - \cos\alpha) \\
&\approx \frac{k^2}{4}\left(3\left(\frac{1}{k}\right) - \left(\frac{1}{k}\right)\left(1 - \frac{1}{2k^2}\right)\right) + k^2\left(1 - \left(1 - \frac{1}{2k^2}\right)\right) \\
&= \frac{k^2}{4}\left(\frac{2}{k} + \frac{1}{2k^3} + 1\right) \\
&\approx \frac{k}{2},
\end{aligned}$$

and

$$\begin{aligned}
[B] &= \frac{1}{4} \cdot \left(2k^2\pi + \pi - 2k^2\alpha - \alpha - 4k\sin\alpha - \sin\alpha\cos\alpha\right) \\
&= \frac{1}{4} \cdot \left(2k^2\pi + \pi - 2k^2 \cdot \frac{1}{k} - \frac{1}{k} - 4k \cdot \frac{1}{k} - \frac{1}{k} \cdot \left(1 - \frac{1}{2k^2}\right)\right) \\
&= \frac{1}{4} \cdot \left(2k^2\pi + \pi - 2k - \frac{1}{k} - 4 - \frac{1}{k} + \frac{1}{2k^3}\right) \\
&\approx \frac{k^2\pi}{2}.
\end{aligned}$$

Therefore,

$$R = [A] + [B] \approx \frac{k^2\pi}{2}$$

Hence,

$$\begin{aligned}\lim_{k\to\infty} \frac{R}{T} &= \lim_{k\to\infty} \frac{\frac{k^2\pi}{2}}{\frac{\pi(2k^2+1)}{4}} \\ &= \lim_{k\to\infty} \frac{2k^2}{2k^2+1} \\ &= 1\end{aligned}$$

as desired.

Similarly, $S$ is given by

$$\begin{aligned}S &= \frac{1}{2} \cdot \int_0^\pi (k(1+\sin\theta))^2 \,\mathrm{d}\theta \\ &= \frac{k^2}{2} \cdot \left[\frac{3}{2} \cdot \theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^\pi \\ &= \frac{k^2}{4}\left(3\pi - \sin\pi\cos\pi\right) + k^2\left(1 - \cos\pi\right) \\ &= \frac{k^2}{4} \cdot 3\pi + 2k^2 \\ &= \left(2 + \frac{3\pi}{4}\right) k^2.\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{k\to\infty} \frac{R}{S} &= \lim_{k\to\infty} \frac{\frac{k^2\pi}{2}}{\left(2+\frac{3\pi}{4}\right)k^2} \\ &= \lim_{k\to\infty} \frac{4\pi}{8+3\pi}.\end{aligned}$$