2023.3 Question 12

1. There are $\binom{2n}{2k}$ ways to select the socks in total.

   All $2k$ socks must be from different pairs of socks, so we have to select $2k$ pairs of socks from the n pairs available, giving us $\binom{n}{2k}$ options.

   Out of those $2k$ pairs, one of the two is selected, which gives 2^{2k}.

   Therefore, the probability is given by

$$\mathrm{P} = \frac{\binom{n}{2k} \cdot 2^{2k}}{\binom{2n}{2k}}.$$

2. There are r pairs of socks, and $2k - 2r = 2(k - r)$ socks that do not form any pairs (single).

   This gives us $\binom{n}{r}$ to select the r pairs of socks, $\binom{n-r}{2(k-r)}$ to select the $2(k - r)$ pairs from the remaining $n - r$ pairs. Finally, there is a factor of $2^{2(k-r)}$ ways to select one sock out of the $n - r$ pair.

   Hence,

$$\mathrm{P}(X_{n,k} = r) = \frac{\binom{n}{r}\binom{n-r}{2(k-r)}2^{2(k-r)}}{\binom{2n}{2k}}$$

   as desired, for $0 \leq r \leq k$.

3. By expanding out the binomial coefficients, we have

$$\begin{aligned}\mathrm{P}(X_{n,k} = r) &= \frac{\frac{n!}{(n-r)!r!} \cdot \frac{(n-r)!}{(2(k-r))!((n-r)-2(k-r))!}}{\frac{(2n)!}{(2k)!(2(n-k))!}} \cdot 2^{2(k-r)} \\ &= \frac{n!(2k)!(2(n-k))!}{(2n)!r!(2(k-r))!(n+r-2k)!} \cdot 2^{2(k-r)},\end{aligned}$$

   and hence

$$\begin{aligned}&\mathrm{P}(X_{n-1,k-1} = r - 1) \\ = &\frac{(n-1)!(2(k-1))!(2((n-1)-(k-1)))!}{(2(n-1))!(r-1)!(2((k-1)-(r-1)))!((n-1)+(r-1)-2(k-1))!} \cdot 2^{2((k-1)-(r-1))} \\ = &\frac{(n-1)!(2k-2)!(2(n-k))!}{(2n-2)!(r-1)!(2(k-r))!(n+r-2k)!} \cdot 2^{2(k-r)}.\end{aligned}$$

   To show that

$$r \cdot \mathrm{P}(X_{n,k} = r) = \frac{k(2k-1)}{2n-1} \cdot \mathrm{P}(X_{n-1,k-1} = r - 1),$$

   it is equivalent to showing that

$$\begin{aligned} r \cdot \frac{n!(2k)!}{(2n)!r!} &= \frac{k(2k-1)}{2n-1} \cdot \frac{(n-1)!(2k-2)!}{(2n-2)!(r-1)!} \\ r \cdot \frac{n(2k)(2k-1)}{(2n)(2n-1)r} &= \frac{k(2k-1)}{2n-1} \\ \frac{n(2k)}{2n} &= k \\ 2nk &= 2nk \end{aligned}$$

   which is true.

   Therefore, we have

$$r \cdot \mathrm{P}(X_{n,k} = r) = \frac{k(2k-1)}{2n-1} \cdot \mathrm{P}(X_{n-1,k-1} = r - 1)$$

   as desired.

Therefore, the expectation can be simplified as

$$\begin{aligned}
\mathrm{E}(X_{n,k}) &= \sum_{r=0}^{k} r\,\mathrm{P}(X_{n,k} = r) \\
&= \sum_{r=1}^{k} r\,\mathrm{P}(X_{n,k} = r) \\
&= \sum_{r=1}^{k} \frac{k(2k-1)}{2n-1}\,\mathrm{P}(X_{n-1,k-1} = r-1) \\
&= \frac{k(2k-1)}{2n-1} \sum_{r=1}^{k} \mathrm{P}(X_{n-1,k-1} = r-1) \\
&= \frac{k(2k-1)}{2n-1} \sum_{r=0}^{k-1} \mathrm{P}(X_{n-1,k-1} = r-1) \\
&= \frac{k(2k-1)}{2n-1} \cdot 1 \\
&= \frac{k(2k-1)}{2n-1}
\end{aligned}$$

since $0 \leq X_{n,k} \leq k$, $0 \leq X_{n-1,k-1} \leq k-1$ and that they can only take integer values.