2023.3 Question 11

$$\begin{aligned}\sum_{k=1}^{N} \frac{k+1}{k!} \cdot x^k &= \sum_{k=1}^{N} \frac{k}{k!} \cdot x^k + \sum_{k=1}^{N} \frac{x^k}{k!} \\ &= \sum_{k=1}^{N} \frac{1}{(k-1)!} \cdot x^k + \sum_{k=0}^{N} \frac{x^k}{k!} - \frac{x^0}{0!} \\ &= \sum_{k=0}^{N-1} \frac{1}{k!} \cdot x^{k+1} + \sum_{k=0}^{N} \frac{x^k}{k!} - 1 \\ &= x \sum_{k=0}^{N-1} \frac{x^k}{k!} + \sum_{k=0}^{N} \frac{x^k}{k!} - 1.\end{aligned}$$

We let $N \to \infty$. Using the Maclaurin Expansion for e^x, we have

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x,$$

and hence

$$\sum_{k=1}^{\infty} \frac{k+1}{k!} \cdot x^k = xe^x + e^x - 1 = (x+1)e^x - 1.$$

1. We have $Y \sim \mathrm{Po}(n)$. Let X_k be the outcome of a k-sided die, i.e. $X_k \sim \mathrm{U}(k)$. WE must have $1 \le X_k \le k$. The random variable D can be defined as

$$D = \begin{cases} 0, & Y = 0, \\ X_k, & Y = k. \end{cases}$$

(a)

$$\begin{aligned}\mathrm{P}(D = 0) &= \mathrm{P}(Y = 0) \\ &= e^{-n} \cdot \frac{n^0}{0!} \\ &= e^{-n}.\end{aligned}$$

(b) For $d \ge 1$, we have

$$\begin{aligned}\mathrm{P}(D = d) &= \sum_{k=d}^{\infty} \mathrm{P}(X_k = d, Y = k) \\ &= \sum_{k=d}^{\infty} \mathrm{P}(X_k = d)\, \mathrm{P}(Y = k) \\ &= \sum_{k=d}^{\infty} \frac{1}{k} \cdot e^{-n} \cdot \frac{n^k}{k!} \\ &= \sum_{k=d}^{\infty} \left(\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n} \right).\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{E}(D) &= \sum_{d=0}^{\infty} d\, \mathrm{P}(D = d) \\ &= \sum_{d=1}^{\infty} d\, \mathrm{P}(D = d) \\ &= \sum_{d=1}^{\infty} \left[d \sum_{k=d}^{\infty} \left(\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n} \right) \right].\end{aligned}$$

This summation is for

$$d \cdot \left(\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n}\right)$$

over the set

$$\begin{aligned}(d, k) \in \{(n, m) \mid n \geq 1, m \geq n\} \\ = \{(n, m) \mid 1 \leq n \leq m\} \\ = \{(n, m) \mid m \geq 1, n \leq m\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{E}(D) &= \sum_{d=1}^{\infty} \left[d \sum_{k=d}^{\infty} \left(\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n}\right)\right] \\ &= \sum_{(d,k) \in \{(n,m) \mid n \geq 1, m \geq n\}} d \cdot \left(\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n}\right) \\ &= \sum_{(d,k) \in \{(n,m) \mid m \geq 1, n \leq m\}} d \cdot \left(\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n}\right) \\ &= \sum_{k=1}^{\infty} \sum_{d=1}^{k} d \cdot \left(\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n}\right) \\ &= \sum_{k=1}^{\infty} \left[\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n} \cdot \sum_{d=1}^{k} d\right].\end{aligned}$$

(c)

$$\begin{aligned}\mathrm{E}(D) &= \sum_{k=1}^{\infty} \left[\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n} \cdot \sum_{d=1}^{k} d\right] \\ &= \sum_{k=1}^{\infty} \left[\frac{1}{k} \cdot \frac{n^k}{k!} \cdot e^{-n} \cdot \frac{k(k+1)}{2}\right] \\ &= \frac{e^{-n}}{2} \sum_{k=1}^{\infty} \frac{n^k(k+1)}{k!} \\ &= \frac{e^{-n}}{2} \left[(n+1) \cdot e^n - 1\right] \\ &= \frac{1}{2} \left[e^{-n} \cdot (n+1) \cdot e^n - e^{-n}\right] \\ &= \frac{1}{2} \left[(n+1) - e^{-n}\right]\end{aligned}$$

as desired.

2. $X_k \sim \mathrm{Po}(k)$ for $k = 1, 2, \cdots, n$. Let Y_n be the outcome of an n-sided die, i.e. $Y_n \sim \mathrm{U}(n)$. Therefore, $Z = X_k$ if $Y_n = k$.

(a) We have

$$\begin{aligned}
\mathrm{P}(Z=0) &= \sum_{k=1}^{n} \mathrm{P}(X_k=0, Y_n=k) \\
&= \sum_{k=1}^{n} \mathrm{P}(X_k=0)\,\mathrm{P}(Y_n=k) \\
&= \sum_{k=1}^{n} e^{-k} \cdot \frac{k^0}{0!} \cdot \frac{1}{n} \\
&= \frac{1}{n} \cdot \sum_{k=1}^{n} e^{-k} \\
&= \frac{1}{n} \cdot \frac{1-\left(e^{-1}\right)^n}{1-e^{-1}} \cdot e^{-1} \\
&= \frac{e^{-1}}{n} \cdot \frac{1-e^{-n}}{1-e^{-1}}.
\end{aligned}$$

(b) For $z \geq 1$, we have

$$\begin{aligned}
\mathrm{P}(Z=z) &= \sum_{k=1}^{n} \mathrm{P}(X_k=z, Y_n=k) \\
&= \sum_{k=1}^{n} \mathrm{P}(X_k=z)\,\mathrm{P}(Y_n=k) \\
&= \frac{1}{n} \cdot \sum_{k=1}^{n} e^{-k} \cdot \frac{k^z}{z!} \\
&= \frac{1}{nz!} \sum_{k=1}^{n} e^{-k} k^z.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{E}(Z) &= \sum_{z=0}^{\infty} z\,\mathrm{P}(Z=z) \\
&= \sum_{z=1}^{\infty} z\,\mathrm{P}(Z=z) \\
&= \sum_{z=1}^{\infty} \left[\frac{1}{n(z-1)!} \cdot \sum_{k=1}^{n} e^{-k} \cdot k^{Z}\right] \\
&= \frac{1}{n} \sum_{z=1}^{\infty} \left[\frac{1}{(z-1)!} \sum_{k=1}^{n} e^{-k} \cdot k^{z}\right] \\
&= \frac{1}{n} \sum_{z=1}^{\infty} \sum_{k=1}^{n} \left(\frac{1}{(z-1)!} \cdot e^{-k} \cdot k^{z}\right) \\
&= \frac{1}{n} \sum_{k=1}^{n} \sum_{z=1}^{\infty} \left(\frac{1}{(z-1)!} \cdot e^{-k} \cdot k^{z}\right) \\
&= \frac{1}{n} \sum_{k=1}^{n} \left[e^{-k} \cdot k \cdot \sum_{z=1}^{\infty} \frac{k^{z-1}}{(z-1)!}\right] \\
&= \frac{1}{n} \sum_{k=1}^{n} \left[e^{-k} \cdot k \cdot \sum_{z=0}^{\infty} \frac{k^{z}}{z!}\right] \\
&= \frac{1}{n} \sum_{k=1}^{n} \left[e^{-k} \cdot k \cdot e^{k}\right] \\
&= \frac{1}{n} \sum_{k=1}^{n} k \\
&= \frac{1}{n} \cdot \frac{n(n+1)}{2} \\
&= \frac{n+1}{2}.
\end{aligned}$$

Therefore, subtracting gives us

$$\begin{aligned}
\mathrm{E}(Z) - \mathrm{E}(D) &= \frac{n+1}{2} - \frac{1}{2} \cdot \left(n + 1 - e^{-n}\right) \\
&= \frac{1}{2} e^{-n} \\
&> 0.
\end{aligned}$$

Therefore, $\mathrm{E}(Z) > \mathrm{E}(D)$ as desired.