2023.3 Question 1

1. The line through P and Q has gradient

$$\frac{aq^2 - ap^2}{2aq - 2ap} = \frac{q^2 - p^2}{2(q-p)} = \frac{p+q}{2},$$

and so it has equation

$$\begin{aligned} y - ap^2 &= \frac{1}{2}(p+q)(x - 2ap) \\ y &= \frac{1}{2}(p+q)x + ap^2 - ap^2 - apq \\ y &= \frac{1}{2}(p+q)x - apq. \end{aligned}$$

The line is tangent to the circle with centre $(0, 3a)$ and radius $2a$, if and only if its distance from $(0, 3a)$ is $2a$.

The line has equation

$$2y - (p+q)x + 2apq = 0$$

and hence the distance is

$$\begin{aligned} d &= \frac{|2 \cdot 3a - (p+q) \cdot 0 + 2apq|}{\sqrt{2^2 + (p+q)^2}} \\ &= \frac{|6a + 2apq|}{\sqrt{4 + p^2q^2 + 6pq + 5}} \\ &= \frac{|2a(3+pq)|}{\sqrt{(pq+3)^2}} \\ &= \frac{2a|3+pq|}{|3+pq|} \\ &= 2a, \end{aligned}$$

and so the distance from l to $(0, 3a)$ is $2a$ as desired.

2. We rearrange the condition to an equation in q

$$\begin{aligned} p^2 + 2pq + q^2 &= p^2q^2 + 6pq + 5 \\ \left(p^2 - 1\right) q^2 + 4pq + \left(5 - p^2\right) &= 0, \end{aligned}$$

and since $p^2 \neq 1$, this must be a quadratic.

We examine the discriminant, Δ:

$$\begin{aligned} \Delta &= (4p)^2 - 4\left(p^2 - 1\right)\left(5 - p^2\right) \\ &= 16p^2 - 4\left(-p^4 - 5 + 6p^2\right) \\ &= 4p^4 - 8p^2 + 20 \\ &= 4\left(p^4 - 4p^2 + 5\right) \\ &= 4\left[\left(p^2 - 2\right)^2 + 1\right] \\ &\geq 4 \cdot 1 \\ &= 4 \\ &> 0, \end{aligned}$$

and so $\Delta > 0$, meaning there will be two distinct real values of q satisfying the condition.

By Vieta's Theorem, we have $q_1 + q_2 = -\frac{4p}{p^2-1}$, and $q_1 q_2 = \frac{5-p^2}{p^2-1}$.

3. Notice that

$$(q_1 + q_2)^2 = \frac{16p^2}{(p^2-1)^2},$$

and

$$\begin{aligned} q_1^2q_2^2 + 6q_1q_2 + 5 &= \frac{(5-p^2)^2}{(p^2-1)^2} + \frac{6 \cdot (5-p^2)}{(p^2-1)} + 5 \\ &= \frac{(5-p^2)^2 + 6(5-p^2)(p^2-1) + 5(p^2-1)^2}{(p^2-1)^2} \\ &= \frac{25 - 10p^2 + p^4 - 6p^4 + 36p^2 - 30 + 5p^4 - 10p^2 + 5}{(p^2-1)^2} \\ &= \frac{16p^2}{(p^2-1)^2}, \end{aligned}$$

and so $(q_1 + q_2)^2 = q_1^2q_2^2 + 6q_1q_2 + 5$.

Let $P(2ap, ap^2)$ for some $p \neq 1$, and let the corresponding solutions to the condition be q_1, q_2. Define the points $Q_1(2aq_1, aq_1^2)$ and $Q_2(2aq_2, aq_2^2)$.

The previous part of the question shows that Q_1 and Q_2 exists and are distinct.

The first part ensures that PQ_1 and PQ_2 are tangents to the circle.

But since q_1 and q_2 satisfies the conditions as well, we must have Q_1Q_2 being a tangent to the circle as well.

Hence, triangle PQ_1Q_2 has all vertices on $x^2 = 4ay$, and that all three sides are tangent to the desired circle.