2023.2 Question 8

1. Let the tetrahedron be $OABC$, and let $|OA| = a, |OB| = b, |OC| = c, |BC| = d, |AC| = e, |AB| = f$.

   This tetrahedron is isosceles, if and only if $a = d, b = e$, and $c = f$.

   The perimeter of the face OAB is $a + b + f$, of face OBC is $b + c + d$, of face OAC is $a + c + e$, and of face ABC is $d + e + f$.

   If the tetrahedron is isosceles, $a = d, b = e$ and $c = f$, then all the faces have perimeter $a + b + c$ and are equal.

   If all faces have equal perimeter, then comparing the perimeters of faces OAB, OBC and OAC, $a + f = c + d$, $b + f = c + e$, $b + d = a + e$.

   Hence, $a - d = b - e = c - f$. Let the difference be t, and $a = d + t, b = e + t, c = f + t$.

   Comparing the perimeter of face OAB and face ABC this time, we have $(d + t) + (e + t) = d + e$, which gives $t = 0$.

   Hence, $a = d$, $b = e$, $c = f$, and the tetrahedron is isosceles.

2. Applying the cosine rule in triangle OBC, we have

   $$|\mathbf{a}|^2 = |\mathbf{b}|^2 + |\mathbf{c}|^2 - 2|\mathbf{b}||\mathbf{c}| \cos \angle COB$$

   and using the dot-product formula

   $$\mathbf{b} \cdot \mathbf{c} = |\mathbf{b}||\mathbf{c}| \cos \angle COB,$$

   rearranging gives us

   $$2\mathbf{b} \cdot \mathbf{c} = |\mathbf{b}|^2 + |\mathbf{c}|^2 - |\mathbf{a}|^2.$$

   Similarly, we have

   $$2\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{c}|^2,$$
   $$2\mathbf{a} \cdot \mathbf{c} = |\mathbf{a}|^2 + |\mathbf{c}|^2 - |\mathbf{b}|^2.$$

   Summing these two, we get

   $$\begin{aligned} 2\mathbf{a} \cdot \mathbf{b} + 2\mathbf{a} \cdot \mathbf{c} &= 2|\mathbf{a}|^2 \\ \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} &= |\mathbf{a}|^2 \\ \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= |\mathbf{a}|^2. \end{aligned}$$

3. Let $\mathbf{g}$ be the position vector for G. $|OG| = |\mathbf{g}| = \frac{1}{4}|\mathbf{a} + \mathbf{b} + \mathbf{c}|$.

   Consider the distance between A and G.

   $$\begin{aligned} |AG| &= \left|\overrightarrow{AG}\right| \\ &= |\mathbf{g} - \mathbf{a}| \\ &= \frac{1}{4}|-3\mathbf{a} + \mathbf{b} + \mathbf{c}|. \end{aligned}$$

   We want to show that $|\mathbf{a} + \mathbf{b} + \mathbf{c}| = |-3\mathbf{a} + \mathbf{b} + \mathbf{c}|$. The following are equivalent

   $$\begin{aligned} |\mathbf{a} + \mathbf{b} + \mathbf{c}| &= |-3\mathbf{a} + \mathbf{b} + \mathbf{c}| \\ |\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 &= |-3\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 \\ |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2\mathbf{a} \cdot \mathbf{b} + 2\mathbf{a} \cdot \mathbf{c} + 2\mathbf{b} \cdot \mathbf{c} &= 9|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 - 6\mathbf{a} \cdot \mathbf{b} - 6\mathbf{a} \cdot \mathbf{c} + 2\mathbf{b} \cdot \mathbf{c} \\ 8\mathbf{a} \cdot \mathbf{b} + 8\mathbf{a} \cdot \mathbf{c} &= 8|\mathbf{a}|^2 \\ \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= |\mathbf{a}|^2 \end{aligned}$$

   and this is true from the previous part.

   Hence, $|OG| = |AG|$. By symmetry, $|OG| = |AG| = |BG| = |CG|$ and hence G is equidistant from all four vertices of the tetrahedron.

4. Notice that

$$\begin{aligned}
|\mathbf{a}-\mathbf{b}-\mathbf{c}|^2 &= |\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2-2\mathbf{a}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{c}+2\mathbf{b}\cdot\mathbf{c} \\
&= |\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2-2|\mathbf{a}|^2+\left(|\mathbf{b}|^2+|\mathbf{c}|^2-|\mathbf{a}|^2\right) \\
&= -2|\mathbf{a}|^2+2|\mathbf{b}|^2+2|\mathbf{c}|^2 \\
&= 2\left(|\mathbf{b}|^2+|\mathbf{c}|^2-|\mathbf{a}|^2\right) \\
&= 4\mathbf{b}\cdot\mathbf{c},
\end{aligned}$$

and since the left-hand side is a square, it is non-negative, which means the dot product is non-negative.

Hence, $\cos\angle BOC \geq 0$, which means it must not be obtuse. By symmetry, this means none of the angles are obtuse.

If one of them is a right angle, say $\angle BOC$, then the dot product evaluates to 0, which must mean $|\mathbf{a}-\mathbf{b}-\mathbf{c}| = 0$.

Hence, $\mathbf{a} = \mathbf{b}+\mathbf{c}$, which means A lies in the plane containing O, B, C. This will not be a tetrahedron, and hence no angles can be right angles.