2023.2 Question 7

1. Let $z = a + ib$ and $|z| = \sqrt{a^2 + b^2}$. Let $w = c + id$ and $|w| = \sqrt{c^2 + d^2}$.
   We have $zw = (ac - bd) + (bc + ad)i$, and hence

$$\begin{aligned} |zw| &= \sqrt{(ac - bd)^2 + (bc + ad)^2} \\ &= \sqrt{a^2c^2 + b^2d^2 - 2abcd + b^2c^2 + a^2d^2 + 2abcd} \\ &= \sqrt{a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2} \\ &= \sqrt{\left(a^2 + b^2\right)\left(c^2 + d^2\right)} \\ &= \sqrt{a^2 + b^2}\sqrt{c^2 + d^2} \\ &= |z||w| \end{aligned}$$

   as desired.

2. Let $z = 2 + i$ and $w = 10 + 11i$, we have $|z| = \sqrt{5}$ and $|w| = \sqrt{221}$.
   Multiplying them gives us $zw = (2 \times 10 - 1 \times 11) + (10 \times 1 + 2 \times 11)i = 9 + 32i$.
   We have $|zw| = |z||w|$, and hence $\sqrt{9^2 + 32^2} = \sqrt{5 \times 221}$.
   This means $9^2 + 32^2 = 5 \times 221$, and hence a possible pair is $(h, k) = (9, 32)$.

3. We have $8045 = 5 \times 1609 = \left(1^2 + 2^2\right)\left(40^2 + 3^2\right)$.
   Let $z = 2 + i, w = 3 + 40i$, $zw = (2 \times 3 - 1 \times 40) + (2 \times 40 + 3 \times 1)i = -34 + 83i$.
   Since $|zw| = |z||w|$, we must have

$$\begin{aligned} 34^2 + 83^2 &= \left(1^2 + 2^2\right) \times \left(40^3 + 3^2\right) \\ &= 5 \times 1609 \\ &= 8045, \end{aligned}$$

   and hence $(m, n) = (34, 83)$ is a possible pair of solution.

4. We notice that 36 is a square number, and

$$\begin{aligned} 36 \times 50805 &= 6^2\left(102^2 + 201^2\right) \\ &= 6^2 \cdot 102^2 + 6^2 \cdot 201^2 \\ &= (6 \times 102)^2 + (6 \times 201)^2 \\ &= 612^2 + 1206^2. \end{aligned}$$

   Hence, $(p, q) = (612, 1206)$ is a possible pair of solution.

5. First, we observe that $1002082 = 1002001 + 81 = 1001^2 + 9^2$, and hence similar to the previous part, we have

$$\begin{aligned} 25 \times 1002082 &= 5^2\left(9^2 + 1001^2\right) \\ &= (5 \times 9)^2 + (5 \times 1001)^2 \\ &= 45^2 + 5005^2, \end{aligned}$$

   and $(r, s) = (45, 5005)$ is a possible pair of solution.
   Furthermore, since $1002082 = 1001^2 + 9^2$, and $5^2 = 4^2 + 3^2$, consider $z = 3 + 4i$, $w = 1001 + 9i$, we have

$$\begin{aligned} zw &= (3 \times 1001 - 4 \times 9) + (4 \times 1001 + 3 \times 9)i \\ &= (3003 - 36) + (4004 + 27)i \\ &= 2967 + 4031i, \end{aligned}$$

   and $(r, s) = (2967, 4031)$ is a possible pair of solution since $|zw| = |z||w|$.

Similarly, $z = 4 + 3i$ and $w = 1001 + 9i$ gives

$$\begin{aligned} zw &= (4 \times 1001 - 3 \times 9) + (3 \times 1001 + 4 \times 9)i \\ &= (4004 - 27) + (3003 + 36)i \\ &= 3977 + 3039i, \end{aligned}$$

and therefore $(R, s) = (3039, 3977)$ is another possible pair of solution.

6. We have $109 = 100 + 9 = 10^2 + 3^2$, and let $z = 10 + 3i$, $w = t + ui$, we examine the linear system of equations

$$\begin{cases} 10t - 3u = 1001, \\ 3t + 10u = 6. \end{cases}$$

This solves to $t = 92$ and $u = -27$. But since $(-27)^2 = 27^2$, we must have $(t.u) = (92, 27)$ satisfies the desired equation.