2023.2 Question 6

The base case is when $n = 1$, and we have

$$\text{LHS} = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 0+1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$\text{RHS} = \mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

so LHS = RHS holds for $n = 1$.

Assume that for some $n = k \geq 1$, the original statement is true.

For $n = k + 1$, we have

$$\begin{aligned}
\text{LHS} &= \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} \\
&= \begin{pmatrix} F_k + F_{k-1} & F_{k-1} + F_{k-2} \\ F_k & F_{k-1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_k & F_{k-1} \\ F_{k-1} & F_{k-2} \end{pmatrix} \\
&= \mathbf{Q} \cdot \mathbf{Q}^k \\
&= \mathbf{Q}^{k+1} \\
&= \text{RHS}.
\end{aligned}$$

So, the original statement holds for $n = 1$ base case, and assuming it holds for some $n = k \geq 1$, it holds for $n = k + 1$. Hence, by the principle of mathematical induction, for all positive integers n, we have

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \mathbf{Q}^n.$$

1. We have

$$\det \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = F_{n+1}F_{n-1} - F_n^2,$$

and on the other hand

$$\begin{aligned}
\det \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} &= \det(\mathbf{Q}^n) \\
&= \det(\mathbf{Q})^n \\
&= (1 \times 0 - 1 \times 1)^n \\
&= (-1)^n.
\end{aligned}$$

Hence,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

for all positive integers n.

2. On one hand,

$$\mathbf{Q}^{m+n} = \begin{pmatrix} F_{m+n+1} & F_{m+n} \\ F_{m+n} & F_{m+n-1} \end{pmatrix},$$

and on the other hand,

$$\begin{aligned}
\mathbf{Q}^{m+n} &= \mathbf{Q}^m \cdot \mathbf{Q}^n \\
&= \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.
\end{aligned}$$

By comparing the top-right entry, we have $F_{m+n} = F_{m+1}F_n + F_mF_{n-1}$ for all positive integers m and n.

3.

$$\begin{aligned}
\text{LHS} &= \mathbf{Q}^2 \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \mathbf{I} + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \mathbf{I} + \mathbf{Q} \\
&= \text{RHS}
\end{aligned}$$

as desired.

(a) On one hand, we have

$$\begin{aligned}
(\mathbf{I} + \mathbf{Q})^n &= \sum_{k=0}^{n} \binom{n}{k} \mathbf{Q}^k \\
&= \sum_{k=0}^{n} \binom{n}{k} \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix},
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
(\mathbf{I} + \mathbf{Q})^n &= \left(\mathbf{Q}^2\right)^n \\
&= \mathbf{Q}^{2n} \\
&= \begin{pmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{pmatrix}.
\end{aligned}$$

Hence, comparing the top-right entry gives us

$$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k$$

as desired.

(b) Notice that,

$$\begin{aligned}
\mathbf{Q}^3 &= \mathbf{Q} \cdot \mathbf{Q}^2 \\
&= \mathbf{Q} (\mathbf{I} + \mathbf{Q}) \\
&= \mathbf{Q} + \mathbf{Q}^2 \\
&= \mathbf{Q} + (\mathbf{I} + \mathbf{Q}) \\
&= \mathbf{I} + 2\mathbf{Q}.
\end{aligned}$$

Hence, on one hand, we have

$$\begin{aligned}
(\mathbf{I} + 2\mathbf{Q})^n &= \sum_{k=0}^{n} \binom{n}{k} (2\mathbf{Q})^k \\
&= \sum_{k=0}^{n} \binom{n}{k} 2^k \mathbf{Q}^k \\
&= \sum_{k=0}^{n} \binom{n}{k} 2^k \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix},
\end{aligned}$$

and on the other hand,

$$\begin{aligned}(\mathbf{I}+2\mathbf{Q})^n &= \left(\mathbf{Q}^3\right)^n \\ &= \mathbf{Q}^{3n} \\ &= \begin{pmatrix} F_{3n+1} & F_{3n} \\ F_{3n} & F_{3n-1} \end{pmatrix}.\end{aligned}$$

Comparing the top-right entry gives us

$$F_{3n} = \sum_{k=0}^{n} \binom{n}{k} 2^k F_k.$$

Also,

$$\begin{aligned}\mathbf{Q}^{3n} &= \mathbf{Q}^n \cdot \mathbf{Q}^{2n} \\ &= \mathbf{Q}^n \sum_{k=0}^{n} \binom{n}{k} \mathbf{Q}^k \\ &= \sum_{k=0}^{n} \binom{n}{k} \mathbf{Q}^{n+k}.\end{aligned}$$

Hence,

$$\begin{pmatrix} F_{3n+1} & F_{3n} \\ F_{3n} & F_{3n-1} \end{pmatrix} = \sum_{k=0}^{n} \binom{n}{k} \begin{pmatrix} F_{n+k+1} & F_{n+k} \\ F_{n+k} & F_{n+k-1} \end{pmatrix},$$

and comparing the top-right entry gives us

$$F_{3n} = \sum_{k=0}^{n} \binom{n}{k} F_{n+k}$$

as desired.

(c) Consider $\mathbf{P} = \mathbf{I} - \mathbf{Q}$, we have

$$\mathbf{P} = \mathbf{I} - \mathbf{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} F_0 & -F_1 \\ -F_1 & F_2 \end{pmatrix}.$$

We experiment $\mathbf{P}^n$ for small ns.

$$\begin{aligned}\mathbf{P}^2 &= \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \\ \mathbf{P}^3 &= \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}, \\ \mathbf{P}^4 &= \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}.\end{aligned}$$

We claim that

$$\mathbf{P}^n = \begin{pmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{pmatrix}$$

and we aim to show this by induction on n.

The base case where $n = 1$ is already shown above. Assume that this statement is true for some $n = k \geq 1$, for $n = k + 1$,

$$\begin{aligned}
\text{LHS} &= \mathbf{P}^{k+1} \\
&= \mathbf{P} \cdot \mathbf{P}^k \\
&= \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} F_{k-1} & -F_k \\ -F_k & F_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} F_k & -F_{k+1} \\ -F_{k-1} - F_k & F_k + F_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} F_k & -F_{k+1} \\ -F_{k+1} & F_{k+2} \end{pmatrix} \\
&= \text{RHS}.
\end{aligned}$$

So the claim is true for the base case $n = 1$. Given it is true for some $n = k$, it is true for $n = k + 1$. Hence, by the principle of mathematical induction, this statement is true for all positive integers n.
This means, we have

$$(\mathbf{I} - \mathbf{Q})^n = \begin{pmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{pmatrix},$$

and hence

$$\begin{aligned}
\mathbf{Q}^n (\mathbf{I} - \mathbf{Q})^n &= \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{pmatrix} \\
&= \begin{pmatrix} F_{n+1}F_{n-1} - F_n^2 & \\ & F_{n+1}F_{n-1} - F_n^2 \end{pmatrix} \\
&= (-1)^n \mathbf{I}.
\end{aligned}$$

On the other hand, using the binomial theorem, we also have

$$\begin{aligned}
\mathbf{Q}^n (\mathbf{I} - \mathbf{Q})^n &= \mathbf{Q}^n \sum_{k=0}^{n} \binom{n}{k} (-\mathbf{Q})^k \\
&= \mathbf{Q}^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k \mathbf{Q}^k \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \mathbf{Q}^{n+k},
\end{aligned}$$

and so

$$(-1)^n \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \begin{pmatrix} F_{n+k+1} & F_{n+k} \\ F_{n+k} & F_{n+k-1} \end{pmatrix}.$$

By comparing the top-right entry, we have

$$\begin{aligned}
0 &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{n+k} \\
(-1)^n \cdot 0 &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n+k} F_{n+k} \\
0 &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n+k} F_{n+k}
\end{aligned}$$

as desired.