2023.2 Question 5

1. (a) By rearranging, we have

$$x_{n+1} = 1 + \frac{1}{x_n + 1},$$

and $x_n \geq 1$ for $n = 0$.

If $x_n \geq 1$ for some $n = k \geq 0$, we must have $\frac{1}{x_k+1} > 0$, and hence

$$x_{k+1} = 1 + \frac{1}{x_k + 1} > 1,$$

and so $x_{k+1} \geq 1$.

Hence, by the principle of mathematical induction, $x_n \geq 1$ for all $n \in \mathbb{N}$.

(b) We have

$$\begin{aligned} x_{n+1}^2 - 2 &= \left(1 + \frac{1}{x_n + 1}\right)^2 - 2 \\ &= 1 + \frac{2}{x_n + 1} + \frac{1}{(x_n+1)^2} - 2 \\ &= \frac{1 + 2(x_n+1) - (x_n+1)^2}{(x_n+1)^2} \\ &= \frac{1 + 2x_n + 2 - x_n^2 - 2x_n - 1}{(x_n+1)^2} \\ &= \frac{-x_n^2 + 2}{(x_n+1)^2} \\ &= -\frac{x_n^2 - 2}{(x_n+1)^2}. \end{aligned}$$

Since

$$\frac{1}{(x_n+1)^2} > 0,$$

it must be $x_{n+1}^2 - 2$ and $x_n^2 - 2$ must take opposite signs.

$$\begin{aligned} \left|x_{n+1}^2 - 2\right| &= \frac{1}{(x_n+1)^2}\left|x_n^2 - 2\right| \\ &\leq \frac{1}{(1+1)^2}\left|x_n^2 - 2\right| \\ &= \frac{1}{4}\left|x_n^2 - 2\right|. \end{aligned}$$

(c) $x_0^2 - 2 = -1 < 0$, and so $x_n^2 - 2 < 0$ for all even n, and > 0 for all odd n.

Hence, $x_{10}^2 - 2 < 0$,and hence $x_{10}^2 \leq 2$.

We have

$$\begin{aligned} \left|x_0^2 - 2\right| &= |1 - 2| = 1, \\ \left|x_1^2 - 2\right| &\leq \frac{1}{4}\left|x_0^2 - 2\right| = \frac{1}{4}, \\ &\vdots \\ \left|x_n^2 - 2\right| &\leq \frac{1}{4^n}, \end{aligned}$$

and hence

$$\left|x_1 0^2 - 2\right| \leq \frac{1}{4^{10}} = \frac{1}{2^{20}}.$$

We have

$$\begin{aligned}2^{20} &= \left(2^{20}\right)^2 \\ &= 1024^2 \\ &> \left(10^3\right)^2 \\ &= 10^6,\end{aligned}$$

and so

$$\left|x_{10}^2 - 2\right| = 2 - x_{10}^2 < 10^{-6},$$

and hence

$$2 - 10^{-6} < x_{10}^2 < 2,$$

which gives

$$2 - 10^{-6} \leq x_{10}^2 \leq 2.$$

2. (a) We have

$$\begin{aligned}y_{n+1} - \sqrt{2} &= \frac{y_n^2 - 2\sqrt{2}y_n}{2y_n} \\ &= \frac{y_n^2 - 2\sqrt{2}y_n + \left(\sqrt{2}\right)^2}{2y_n} \\ &= \frac{\left(y_n - \sqrt{2}\right)^2}{2y_n}.\end{aligned}$$

$y_n \geq 1$ is true for the base case $n = 0$.
If it is true for $n = k$, we have

$$\frac{\left(y_n - \sqrt{2}\right)^2}{2y_n} \geq 0,$$

and so $y_{n+1} - \sqrt{2} \geq 0$, and hence $y_{n+1} \geq \sqrt{2} \geq 1$ as desired.
In fact, we can conclude that $y_n \geq \sqrt{2}$ for all $n \geq 1$.

(b) Since $y_n \geq 1$, we have $0 \leq \frac{1}{y_n} \leq 1$, and hence we have

$$y_{n+1} - \sqrt{2} \leq \frac{\left(y_n - \sqrt{2}\right)^2}{2}.$$

We aim to show the desired result by induction on n. The base case when $n = 1$ is

$$y_1 = \frac{y_0^2 + 2}{2y_0} = \frac{1^2 + 2}{2} = \frac{3}{2},$$

and

$$\text{RHS} = 2 \cdot \left(\frac{\sqrt{2} - 1}{2}\right)^{2 \cdot 1} = 2 \cdot \frac{2 + 1 - 2\sqrt{2}}{4} = \frac{3}{2} - \sqrt{2},$$

and hence

$$\text{LHS} = y_1 - \sqrt{2} = \frac{3}{2} - \sqrt{2} \leq \text{RHS}$$

as desired.

Now we assume the desired result is true for some $n = k$. For $n = k + 1$,

$$\begin{aligned}
y_{k+1} - \sqrt{2} &\leq \frac{\left(y_k - \sqrt{2}\right)^2}{2} \\
&\leq \frac{\left[2 \cdot \left(\frac{\sqrt{2}-1}{2}\right)^{2^k}\right]^2}{2} \\
&= \frac{4 \cdot \left(\frac{\sqrt{2}-1}{2}\right)^{2^k \cdot 2}}{2} \\
&= 2 \cdot \left(\frac{\sqrt{2}-1}{2}\right)^{2^{k+1}},
\end{aligned}$$

which is precisely the desired statement for $n = k + 1$.

So the desired is true for the base case where $n = 1$. Given it is true for some $n = k$, it is true for $n = k + 1$. Hence, by the principle of mathematical induction,

$$y_n - \sqrt{2} \leq 2 \cdot \left(\frac{\sqrt{2}-1}{2}\right)^{2^n}$$

for all $n \geq 1$.

(c) First, we have $y_{10} \geq \sqrt{2}$ by the stronger bound found for the first part.
Additionally,

$$\begin{aligned}
y_{10} - \sqrt{2} &\leq 2 \cdot \left(\frac{\sqrt{2}-1}{2}\right)^{2^{10}} \\
&\leq 2 \cdot \left(\frac{\frac{1}{2}}{2}\right)^{2^{10}} \\
&= 2 \cdot \left(\frac{1}{2^2}\right)^{2^{10}} \\
&= 2 \cdot \left(\frac{1}{2}\right)^{2^{10} \cdot 2} \\
&= \frac{2}{2^{2^{11}}} \\
&= \frac{1}{2^{2^{11}-1}}.
\end{aligned}$$

For the bound, notice that

$$\begin{aligned}
\frac{1}{2^{2^{11}-1}} &= \frac{1}{2^{2048-1}} \\
&= \frac{1}{2^{2047}} \\
&< \frac{1}{2^{2040}} \\
&= \frac{1}{\left(2^{10}\right)^{204}} \\
&< \frac{1}{\left(10^3\right)^{204}} \\
&< \frac{1}{\left(10^3\right)^{200}} \\
&= \frac{1}{10^{600}},
\end{aligned}$$

and so

$$y_{10} \leq \sqrt{2} + 10^{-600}.$$

Hence, we can conclude

$$\sqrt{2} \leq y_{10} \leq \sqrt{2} + 10^{-600}$$

as desired.