2023.2 Question 4

1. We have

$$\begin{aligned}\left(x-\sqrt{2}\right)^2&=3\\ x^2-2\sqrt{2}x+2&=3\\ x^2-1&=2\sqrt{2}x\\ x^4-2x^2+1&=8x^2\\ x^4-10x^2+1&=0\end{aligned}$$

as desired.

If $f(x)=x^4-10x^2+1$, we notice that $x=\sqrt{2}+\sqrt{3}$ satisfies $\left(x-\sqrt{2}\right)^2=\left(\sqrt{3}\right)^2=3$, and hence $f\left(\sqrt{2}+\sqrt{3}\right)=0$ as desired.

2. We have

$$\begin{aligned}\left(x-\left(\sqrt{2}+\sqrt{3}\right)\right)^2&=\left(\sqrt{5}\right)^2=5\\ x^2-2\left(\sqrt{2}+\sqrt{3}\right)x+2+3+2\sqrt{6}&=5\\ x^2+2\sqrt{6}&=2\left(\sqrt{2}+\sqrt{3}\right)x\\ x^4+2\cdot2\sqrt{6}\cdot x^2+\left(2\sqrt{6}\right)^2&=4\left(\sqrt{2}+\sqrt{3}\right)^2x^2\\ x^4+4\sqrt{6}x^2+24&=4\left(5+2\sqrt{6}\right)x^2\\ x^4+4\sqrt{6}x^2+24&=20x^2+8\sqrt{6}x^2\\ x^4-20x^2+24&=4\sqrt{6}x^2\\ \left(x^4-20x^2+24\right)^2&=\left(4\sqrt{6}x^2\right)^2\\ x^8-40x^6+448x^4-960x^2+576&=96x^4\\ x^8-40x^6+352x^4-960x^2+576&=0.\end{aligned}$$

Therefore, the polynomial

$$g(x)=x^8-40x^6+352x^4-960x^2+576$$

satisfies $g\left(\sqrt{2}+\sqrt{3}+\sqrt{5}\right)=0$ as desired.

3. If $t=a,b,c$ are solutions to the cubic equation $t^3-3t+1=0$ in t, then $t=a+\sqrt{2},b+\sqrt{2},c+\sqrt{2}$ are solutions to the cubic equation in t

$$\begin{aligned}\left(y-\sqrt{2}\right)^3-3\left(t-\sqrt{2}\right)+1&=0\\ t^3-3\sqrt{2}t^2+6t-2\sqrt{2}-3t+3\sqrt{2}+1&=0\\ t^3+3t+1&=3\sqrt{2}t^2-\sqrt{2}\\ t^6+6t^4+2t^3+9t^2+6t+1&=18t^4-12t^2+2\\ t^6-12t^4+2t^3+21t^2+6t-1&=0.\end{aligned}$$

Therefore, the polynomial

$$h(x)=x^6-12x^4+2x^3+21x^2+6x-1$$

satisfies $h\left(a+\sqrt{2}\right)=h\left(b+\sqrt{2}\right)=h\left(c+\sqrt{2}\right)=0$ as desired.

4. We have

$$\begin{aligned}
\left(x - \sqrt[3]{2}\right)^3 &= 3 \\
x^3 - 3\sqrt[3]{2}x^2 + 3\sqrt[3]{4}x - 2 &= 3 \\
x^3 - 5 &= 3\sqrt[3]{2}x^2 - 3\sqrt[3]{4}x \\
x^3 - 5 &= 3\sqrt[3]{2}x\left(x - \sqrt[3]{2}\right) \\
x^3 - 5 &= 3\sqrt[3]{2}x \cdot \sqrt[3]{3} \\
x^3 - 5 &= 3\sqrt[3]{6}x \\
x^9 - 15x^6 + 75x^3 - 125 &= 162x^3 \\
x^9 - 15x^6 - 87x^3 - 125 &= 0.
\end{aligned}$$

Therefore, the polynomial

$$k(x) = x^9 - 15x^6 - 87x^3 - 125 = 0$$

satisfies $k\left(\sqrt[3]{2} + \sqrt[3]{3}\right) = 0$.