2023.2 Question 3

1. (a) If n is odd, then p must be negative when either $x \gg 0$ or $x \ll 0$, for a sufficiently large $|x|$, since the leading term (term with x^n) will be sufficiently large at this point. Since $p(x) > 0$, n must be even. Furthermore, the leading term coefficient must be positive.

   For $0 \leq k \leq n$, $p^{(k)}(x)$ is an $n-k$ degree polynomial. Hence, q is also a degree n polynomial, with a positive leading term coefficient. This means when $|x|$ is sufficiently large, the leading term will be sufficiently positive and q will be positive.

   (b) We would like to show that $q(x) - q'(x) = p(x)$, and we have

$$\begin{aligned}
q(x) - q'(x) &= \sum_{k=0}^{n} p^{(k)}(x) - \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{n} p^{(k)}(x) \\
&= \sum_{k=0}^{n} p^{(k)}(x) - \sum_{k=0}^{n} p^{(k+1)}(x) \\
&= \sum_{k=0}^{n} p^{(k)}(x) - \sum_{k=1}^{n+1} p^{(k)}(x) \\
&= p^{(0)}(x) - p^{(n+1)}(x) \\
&= p(x) - 0 \\
&= p(x),
\end{aligned}$$

   as desired.

2. (a) If $q'(x) = 0$ for some x, then $0 = p(x) - q(x)$, giving $p(x) = q(x)$ for that point. This means $p(x)$ and $q(x)$ will meet at that point, proving precisely $p(x)$ and $q(x)$ meet at every stationary point of $y = q(x)$.

   This means q has all local minimums being positive, since they must be stationary points, situated on p as well, being positive.

   Since q is an even-degree polynomial, it must also be the case that one of the local minimums is a global minimum, which is positive.

   Hence, q is always positive, and $q(x) > 0$ for all x.

   (b) By differentiating, we have

$$\begin{aligned}
\frac{\mathrm{d}e^{-x}q(x)}{\mathrm{d}x} &= e^{-x}q'(x) - e^{-x}q(x) \\
&= e^{-x}(q'(x) - q(x)) \\
&= -e^{-x}p(x).
\end{aligned}$$

   We have $e^{-x} > 0$ and $p(x) > 0$ for all x, which means the gradient is always negative, which shows that $e^{-x}q(x)$ is decreasing.

   For sufficiently large x, $q(x) > 0$, and hence $e^{-x}q(x) > 0$ for sufficiently large x.

   Since this function is decreasing, we can conclude that $e^{-x}q(x) > 0$ for all x, and since e^{-x} is always positive, it must be the case that $q(x) > 0$ for all x.

   (c) Let the upper bound of the integral be N. Using integration by parts, we have

$$\begin{aligned}
\int_0^N p(x+t)e^{-t} \,\mathrm{d}t &= -\int_0^N p(x+t) \,\mathrm{d}e^{-t} \\
&= -\left[p(x+t)e^{-t}\right]_0^N + \int_0^N e^{-t} \,\mathrm{d}p(x+t) \\
&= p(x) - p(x+N)e^{-N} + \int_0^N p'(x+t)e^{-t} \,\mathrm{d}t.
\end{aligned}$$

   Let $N \to \infty$, $e^{-N}p(x+N) \to 0$ since an exponential dominates a polynomial. Hence,

$$\int_0^\infty p(x+t)e^{-t} \,\mathrm{d}t = p(x) + \int_0^\infty p^{(1)}(x+t)e^{-t} \,\mathrm{d}t$$

as desired.

Repeating this process, we have

$$\begin{aligned}
\int_0^\infty p(x+t)e^{-t}\,\mathrm{d}t &= p(x) + \int_0^\infty p^{(1)}(x+t)e^{-t}\,\mathrm{d}t \\
&= p(x) + p^{(1)}(x) + \int_0^\infty p^{(2)}(x+t)e^{-t}\,\mathrm{d}t \\
&= \cdots \\
&= p(x) + p^{(1)}(x) + \cdots + p^{(n)}(x) + \int_0^\infty p^{(n+1)}(x+t)e^{-t}\,\mathrm{d}t \\
&= \sum_{k=0}^{n} p^{(k)}(x) + \int_0^\infty 0\,\mathrm{d}t \\
&= q(x) + 0 \\
&= q(x),
\end{aligned}$$

as desired.

Since the integrand of this integral is positive for all $t \geq 0$, the integral must evaluate to a positive value, and hence $q(x) > 0$ for all x as desired.