2023.2 Question 2

1. Let
$$f(t) = \frac{2t}{1-t^2}.$$
By the double angle formula for tan, we have
$$f(\tan\theta) = \tan 2\theta.$$
Since $y = f(x)$, we have $y = f(\tan\alpha) = \tan 2\alpha$. Similarly, $z = f(y) = \tan 4\alpha$, and $x = f(z) = \tan 8\alpha$.
But since $x = x$, we must have $\tan\alpha = \tan 8\alpha$, and there must be some $k \in \mathbb{Z}$ such that
$$\alpha + k\pi = 8\alpha,$$
i.e.
$$\alpha = \frac{k\pi}{7}.$$
Since $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$ for the substitution, we have
$$\alpha = -\frac{3}{7}\pi, -\frac{2}{7}\pi, -\frac{1}{7}\pi, 0, \frac{1}{7}\pi, \frac{2}{7}\pi, \frac{3}{7}\pi,$$
and hence
$$(\alpha, 2\alpha, 4\alpha) = (0,0,0)\,, \left(\pm\frac{1}{7}\pi, \pm\frac{2}{7}\pi, \pm\frac{4}{7}\pi\right), \left(\pm\frac{2}{7}\pi, \pm\frac{4}{7}\pi, \pm\frac{8}{7}\pi\right), \left(\pm\frac{3}{7}\pi, \pm\frac{6}{7}\pi, \pm\frac{12}{7}\pi\right),$$
which means
$$(x,y,z) = (\tan 0, \tan 0, \tan 0)\,,$$
or
$$(x,y,z) = \left(\tan\pm\frac{1}{7}\pi, \tan\pm\frac{2}{7}\pi, \tan\pm\frac{4}{7}\pi\right) = \left(\tan\pm\frac{1}{7}\pi, \tan\pm\frac{2}{7}\pi, \tan\mp\frac{3}{7}\pi\right),$$
or
$$(x,y,z) = \left(\tan\pm\frac{2}{7}\pi, \tan\pm\frac{4}{7}\pi, \tan\pm\frac{8}{7}\pi\right) = \left(\tan\pm\frac{2}{7}\pi, \tan\mp\frac{3}{7}\pi, \tan\pm\frac{1}{7}\pi\right),$$
or
$$(x,y,z) = \left(\tan\pm\frac{3}{7}\pi, \tan\pm\frac{6}{7}\pi, \tan\pm\frac{12}{7}\pi\right) = \left(\tan\pm\frac{3}{7}\pi, \tan\mp\frac{1}{7}\pi, \tan\mp\frac{2}{7}\pi\right).$$

2. Let
$$g(t) = \frac{3t - t^3}{1 - 3t^2}.$$
The triple angle formula for tan is given by
$$\begin{aligned}\tan 3\theta &= \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta} \\ &= \frac{\tan\theta + \frac{2\tan\theta}{1-\tan^2\theta}}{1 - \tan\theta\frac{2\tan\theta}{1-\tan^2\theta}} && = \frac{\tan\theta(1-\tan^2\theta) + 2\tan\theta}{(1-\tan^2\theta) - \tan\theta(2\tan\theta)} \\ &= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta},\end{aligned}$$
and hence
$$g(\tan\theta) = \tan 3\theta.$$
Let $x = \tan\alpha$ for $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$. We must have $y = \tan 3\alpha$, $z = \tan 9\alpha$ and $x = \tan 27\alpha$.
There must exist some $k \in \mathbb{Z}$ such that
$$\alpha + k\pi = 27\alpha,$$

and hence

$$\alpha = \frac{k\pi}{26}.$$

It must be the case that $-13 < k < 13$, and this leads to $-12 \leq k \leq 12$. These all lead to distinct values of x.

We already have $\alpha \neq t\pi + \frac{\pi}{2}$ for any $t \in \mathbb{Z}$.

We still verify that $2\alpha \neq t\pi + \frac{\pi}{2}$. We have that

$$\begin{aligned} 2\alpha - \frac{\pi}{2} &= \frac{k\pi}{13} - \frac{\pi}{2} \\ &= \frac{(2k-13)\pi}{26}. \end{aligned}$$

$2k - 13$ cannot be a multiple of 13 apart from $k = 0$ (in which case it is still not a multiple of 26), hence not of 26, and hence $2\alpha \neq t\pi + \frac{\pi}{2}$.

A similar reasoning applies for 4α:

$$\begin{aligned} 4\alpha - \frac{\pi}{2} &= \frac{2k\pi}{13} - \frac{\pi}{2} \\ &= \frac{(4k-13)\pi}{26}. \end{aligned}$$

$4k - 13$ cannot be a multiple of 13 apart from $k = 0$ (in which case it is still not a multiple of 26), hence not of 26, and hence $4\alpha \neq t\pi + \frac{\pi}{2}$.

Therefore, all 25 values of k leads to pairs of solutions for (x, y, z), and they must all be distinct (since xs) are distinct.

Therefore, there are 25 pairs of distinct real solutions to the simultaneous solutions.

3. (a) Let $h(t) = 2t^2 - 1$. Notice that by the cosine double angle formula,

$$h(\cos\theta) = \cos 2\theta.$$

If $|x|, |y|, |z| \leq 1$, let $x = \cos\alpha$ for $0 \leq \alpha \leq \pi$. We must have $y = \cos 2\alpha, z = \cos 4\alpha$, and $x = \cos 8\alpha$, leading to $\cos\alpha = \cos 8\alpha$.

Hence, we must have, for $k \in \mathbb{Z}$, that

$$8\alpha = 2k\pi \pm \alpha,$$

which gives

$$\alpha = \frac{2k\pi}{7}$$

or

$$\alpha = \frac{2k\pi}{9}.$$

Therefore, we have

$$\alpha = 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{6\pi}{9}, \frac{8\pi}{9}$$

which gives 8 pairs of solutions for (x, y, z).

(b) We have $x = h^3(x)$, and hence x satisfies a polynomial with degree 8. Hence, there are at most 8 distinct real roots for x, and since there are 8 of them for which $|x| \leq 1$, it must be the case that they are all of them. Hence, all solutions to the equations satisfy $|x|, |y|, |z| \leq 1$.