2023.2 Question 12

1. We first consider the event $Y \leq t$.

$$\begin{aligned} Y \leq t &\iff \max\{X_1, X_2, \ldots, X_n\} \leq t \\ &\iff X_1, X_2, \ldots, X_n \leq t \\ &\iff X_1 \leq t, X_2 \leq t, \cdots, X_n \leq t. \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{P}(Y \leq t) &= \mathrm{P}(X_1 \leq t, X_2 \leq t, \cdots, X_n \leq t) \\ &= \mathrm{P}(X_1 \leq t)\,\mathrm{P}(X_2 \leq t) \cdot \mathrm{P}(X_n \leq t) \\ &= [\mathrm{P}(X_1 \leq t)]^n \end{aligned}$$

as desired.

We first find the cumulative distribution function of X, F. For $0 \leq x \leq \pi$,

$$\begin{aligned} F(x) &= \int_0^x f(t)\,\mathrm{d}t \\ &= \frac{1}{2}\int_0^x \sin t\,\mathrm{d}t \\ &= -\frac{1}{2}[\cos t]_0^x \\ &= \frac{1}{2}(1 - \cos x). \end{aligned}$$

Now, let G be the cumulative distribution function of Y. We have $0 \leq Y \leq \pi$. For $0 \leq y \leq \pi$,

$$\begin{aligned} G(y) &= \mathrm{P}(Y \leq y) \\ &= [\mathrm{P}(X_1 \leq y)]^n \\ &= [F(y)]^n \\ &= \left[\frac{1}{2}(1 - \cos y)\right]^n &= \frac{1}{2^n}(1 - \cos y)^n. \end{aligned}$$

Hence, the probability density function of Y, g, is given by

$$\begin{aligned} g(y) &= G'(y) \\ &= \frac{1}{2^n} \cdot n \cdot \sin x \cdot (1 - \cos X)^{n-1} \\ &= \frac{n \sin x\,(1 - \cos X)^{n-1}}{2^n} \end{aligned}$$

for $0 \leq t \leq \pi$, and 0 otherwise.

2. $m(n)$ is such that

$$\begin{aligned} G(m(n)) &= \frac{1}{2} \\ \frac{1}{2^n}(1 - \cos m(n))^n &= \frac{1}{2} \\ (1 - \cos m(n))^n &= 2^{n-1} \\ 1 - \cos m(n) &= 2^{\frac{n-1}{n}} \\ \cos m(n) &= 1 - 2^{1-\frac{1}{n}} \\ m(n) &= \arccos\left(1 - 2^{1-\frac{1}{n}}\right). \end{aligned}$$

As n increases, $\frac{1}{n}$ decreases, $1 - \frac{1}{n}$ increases, $2^{1-\frac{1}{n}}$ increases, $1 - 2^{1-\frac{1}{n}}$ increases, and so $m(n)$ increases. $m(n) \to \pi$ as $n \to \infty$.

3. By definition, we have

$$
\begin{aligned}
\mu(n) &= \mathrm{E}(Y) \\
&= \int_0^\pi \frac{n}{2^n} x \sin x \left(1-\cos x\right)^{n-1} \mathrm{d}x \\
&= \frac{1}{2^n} \int_0^\pi x \cdot n \sin x \left(1-\cos x\right)^{n-1} \mathrm{d}x \\
&= \frac{1}{2^n} \int_0^\pi x \cdot \left(1-\cos x\right)^{n} \mathrm{d}x \\
&= \frac{1}{2^n} \left[x \left(1-\cos x\right)^n\right]_0^\pi - \frac{1}{2^n} \int_0^\pi \left(1-\cos x\right)^n \mathrm{d}x \\
&= \frac{1}{2^n} \left[\pi \cdot (1+1)^n - 0 \cdot (1-1)^n\right] - \frac{1}{2^n} \int_0^\pi \left(1-\cos x\right)^n \mathrm{d}x \\
&= \frac{1}{2^n} \cdot \pi \cdot 2^n - \frac{1}{2^n} \int_0^\pi \left(1-\cos x\right)^n \mathrm{d}x \\
&= \pi - \frac{1}{2^n} \int_0^\pi \left(1-\cos x\right)^n \mathrm{d}x.
\end{aligned}
$$

(a) By taking difference of two consecutive terms of $\mu(n)$, we have

$$
\begin{aligned}
\mu(n+1) - \mu(n) &= \left[\pi - \frac{1}{2^{n+1}} \int_0^\pi \left(1-\cos x\right)^{n+1} \mathrm{d}x\right] - \left[\pi - \frac{1}{2} \int_0^\pi \left(1-\cos x\right)^{n} \mathrm{d}x\right] \\
&= \frac{1}{2^n} \int_0^\pi \left(1-\cos x\right)^n \mathrm{d}x - \frac{1}{2^{n+1}} \int_0^\pi \left(1-\cos x\right)^{n+1} \mathrm{d}x \\
&= \frac{1}{2^{n+1}} \int_0^\pi \left[2\left(1-\cos x\right)^n - \left(1-\cos x\right)^{n+1}\right] \mathrm{d}x \\
&= \frac{1}{2^{n+1}} \int_0^\pi \left(1-\cos x\right)^n \left[2 - (1-\cos x)\right] \mathrm{d}x \\
&= \frac{1}{2^{n+1}} \int_0^\pi \left(1-\cos x\right)^n \left(1+\cos x\right) \mathrm{d}x.
\end{aligned}
$$

For $0 < x < \pi$, we have $0 < \cos x < 1$, and so the integrand is positive on the interval. Hence, $\mu(n+1) - \mu(n) > 0$, and $\mu(n+1) > \mu(n)$, and hence $\mu(n)$ increases with n.

(b) On one hand, we have

$$
m(2) = \arccos\left(1 - 2^{1-\frac{1}{2}}\right) = \arccos\left(1 - \sqrt{2}\right).
$$

On the other hand,

$$
\begin{aligned}
\mu(2) &= \pi - \frac{1}{4} \int_0^\pi \left(1-\cos x\right)^2 \mathrm{d}x \\
&= \pi - \frac{1}{4} \int_0^\pi \left(1 - 2\cos x + \cos^2 x\right) \mathrm{d}x \\
&= \pi - \frac{1}{4} \int_0^\pi \left(1 - 2\cos x + \frac{\cos 2x + 1}{2}\right) \mathrm{d}x \\
&= \pi - \frac{1}{4} \int_0^\pi \left(\frac{3}{2} - 2\cos x + \frac{1}{2}\cos 2x\right) \mathrm{d}x \\
&= \pi - \frac{1}{4} \left(\frac{3}{2}x - 2\sin x + \frac{1}{4}\sin 2x\right)_0^\pi \\
&= \pi - \frac{1}{4} \left[\frac{3}{2}\left(\pi - x\right) - 2\left(\sin\pi - \sin 0\right) + \frac{1}{4}\left(\sin 2\pi - \sin 0\right)\right] \\
&= \pi - \frac{1}{4} \cdot \frac{3}{2}\pi \\
&= \frac{5}{8}\pi.
\end{aligned}
$$

We want to show that

$$\left(0 < \frac{1}{2}\pi <\right) \frac{5}{8}\pi < \arccos\left(1 - \sqrt{2}\right) (< \pi),$$

and this is equivalent to showing that

$$\cos\frac{5}{8}\pi > 1 - \sqrt{2}.$$

We first notice that $\cos\frac{5}{8}\pi = \cos\left(\frac{1}{2}\pi + \frac{1}{8}\pi\right)$, and notice that $\cos\left(\frac{1}{8}\pi\right)$ is such that

$$2\cos^2\left(\frac{1}{8}\pi\right) - 1 = \cos\left(2 \cdot \frac{1}{8}\pi\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

and hence

$$2\cos^2\frac{\pi}{8} = 1 + \frac{1}{\sqrt{2}} = \frac{2+\sqrt{2}}{2},$$

meaning

$$\cos\frac{\pi}{8} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

Therefore,

$$\sin^2\frac{\pi}{8} = 1 - \frac{2+\sqrt{2}}{4} = \frac{2-\sqrt{2}}{4}$$

and hence

$$\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

Hence,

$$\begin{aligned}\cos\frac{5}{8}\pi &= \cos\left(\frac{1}{2}\pi + \frac{1}{8}\pi\right)\\ &= \cos\frac{1}{2}\pi\cos\frac{1}{8}\pi - \sin\frac{1}{2}\pi\sin\frac{1}{8}\pi\\ &= 0 - \sin\frac{1}{8}\pi\\ &= -\frac{\sqrt{2-\sqrt{2}}}{2}.\end{aligned}$$

Finally, we have the following being equivalent:

$$\begin{aligned}\cos\frac{5}{8}\pi &> 1 - \sqrt{2}\\ (0 >) - \frac{\sqrt{2-\sqrt{2}}}{2} &> 1 - \sqrt{2}\\ \sqrt{2} - 1 &> \frac{\sqrt{2-\sqrt{2}}}{2}\\ 2 + 1 - 2\sqrt{2} &> \frac{2-\sqrt{2}}{4}\\ 12 - 8\sqrt{2} &> 2 - \sqrt{2}\\ 7\sqrt{2} &< 10\\ 49 \cdot 2 = 98 &< 100\end{aligned}$$

is true, and hence $\mu(2) < m(2)$ as desired.