2023.2 Question 1

1. If $x = \frac{1}{t}$, we have
$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{t^2}$$
and hence
$$\mathrm{d}x = -\frac{\mathrm{d}t}{t^2}.$$
Hence,
$$\begin{aligned}\int_a^b \frac{\mathrm{d}x}{(1+x^2)^{\frac{3}{2}}} &= \int_{a^{-1}}^{b^{-1}} \frac{-\mathrm{d}t}{t^2\left(1+\frac{1}{t^2}\right)^{\frac{3}{2}}} \\ &= \int_{a^{-1}}^{b^{-1}} \frac{-t\,\mathrm{d}t}{t^3\left(1+\frac{1}{t^2}\right)^{\frac{3}{2}}} \\ &= \int_{a^{-1}}^{b^{-1}} \frac{-t\,\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}}\end{aligned}$$
as desired.

2. We have
$$\int_{a^{-1}}^{b^{-1}} \frac{-t\,\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}} = \left[\left(1+t^2\right)^{-\frac{1}{2}}\right]_{a^{-1}}^{b^{-1}}.$$

   (a)
$$\begin{aligned}\int_{\frac{1}{2}}^{2} \frac{\mathrm{d}x}{(1+x^2)^{\frac{3}{2}}} &= \int_{2}^{\frac{1}{2}} \frac{-t\,\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}} \\ &= \left[\left(1+t^2\right)^{-\frac{1}{2}}\right]_2^{\frac{1}{2}} \\ &= \left(1+\left(\frac{1}{2}\right)^2\right)^{-\frac{1}{2}} - \left(1+(2)^2\right)^{-\frac{1}{2}} \\ &= \frac{1}{\sqrt{\frac{5}{4}}} - \frac{1}{\sqrt{5}} \\ &= \frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}} \\ &= \frac{1}{\sqrt{5}}.\end{aligned}$$

(b) Notice that the integrand is even, we have

$$\begin{aligned}
\int_{-2}^{2} \frac{\mathrm{d}x}{(1+x^2)^{\frac{3}{2}}} &= 2\int_{0}^{2} \frac{\mathrm{d}x}{(1+x^2)^{\frac{3}{2}}} \\
&= 2\lim_{u\to 0^+}\int_{u}^{2} \frac{\mathrm{d}x}{((1+x^2))^{\frac{3}{2}}} \\
&= 2\lim_{u\to 0^+}\int_{\frac{1}{u}}^{\frac{1}{2}} \frac{-t\,\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}} \\
&= 2\lim_{u\to \infty}\int_{u}^{\frac{1}{2}} \frac{-t\,\mathrm{d}t}{(1+t^2)^{\frac{3}{2}}} \\
&= 2\lim_{u\to \infty}\left[\left(1+t^2\right)^{-\frac{1}{2}}\right]_u^2 \\
&= 2\cdot\left(\frac{2}{\sqrt{5}} - \lim_{u\to\infty}\frac{1}{\sqrt{1+u^2}}\right) \\
&= 2\cdot\left(\frac{2}{\sqrt{5}} - 0\right) \\
&= \frac{4}{\sqrt{5}}.
\end{aligned}$$

3. (a) Starting from the left, we have

$$\begin{aligned}
\int_{\frac{1}{2}}^{2} \frac{\mathrm{d}x}{(1+x^2)^2} &= \int_{2}^{\frac{1}{2}} \frac{-\frac{1}{t^2}\,\mathrm{d}t}{\left(1+\frac{1}{t^2}\right)^2} \\
&= \int_{\frac{1}{2}}^{2} \frac{\frac{1}{t^2}\cdot t^4\,\mathrm{d}t}{t^4\left(1+\frac{1}{t^2}\right)^2} \\
&= \int_{\frac{1}{2}}^{2} \frac{t^2\,\mathrm{d}t}{(1+t^2)^2},
\end{aligned}$$

and therefore the first equal sign is true.

As for the second equal sign, we notice that

$$\begin{aligned}
\int_{\frac{1}{2}}^{2} \frac{\mathrm{d}x}{(1+x^2)^2} + \int_{\frac{1}{2}}^{2} \frac{x^2\,\mathrm{d}x}{(1+x^2)^2} &= \int_{\frac{1}{2}}^{2} \frac{\left(1+x^2\right)\mathrm{d}x}{(1+x^2)^2} \\
&= \int_{\frac{1}{2}}^{2} \frac{\mathrm{d}x}{1+x^2},
\end{aligned}$$

which means that

$$\int_{\frac{1}{2}}^{2} \frac{\mathrm{d}x}{(1+x^2)^2} = \int_{\frac{1}{2}}^{2} \frac{x^2\,\mathrm{d}x}{(1+x^2)^2} = \frac{1}{2}\int_{\frac{1}{2}}^{2} \frac{\mathrm{d}x}{1+x^2}.$$

Hence,

$$\begin{aligned}
\int_{\frac{1}{2}}^{2} \frac{\mathrm{d}x}{(1+x^2)^2} &= \frac{1}{2}\int_{\frac{1}{2}}^{2} \frac{\mathrm{d}x}{1+x^2} \\
&= \frac{1}{2}\left[\arctan x\right]_{\frac{1}{2}}^{2} \\
&= \frac{1}{2}\arctan 2 - \frac{1}{2}\arctan\frac{1}{2} \\
&= \frac{1}{2}\arctan 2 - \frac{1}{2}\left(\frac{\pi}{2} - \arctan 2\right) \\
&= \arctan 2 - \frac{\pi}{4}.
\end{aligned}$$

(b) Let $x = \frac{1}{u}$, we have $\mathrm{d}x = -\frac{1}{u^2}\,\mathrm{d}u$.

Let the integral be I, and we have

$$\begin{aligned}
I &= \int_{\frac{1}{2}}^{2} \frac{1-x}{x\left(1+x^2\right)^{\frac{1}{2}}} \, \mathrm{d}x \\
&= \int_{\frac{1}{2}}^{2} \frac{1-\frac{1}{u}}{\frac{1}{u}\left(1+\frac{1}{u^2}\right)^{\frac{1}{2}}} \cdot \frac{1}{u^2} \, \mathrm{d}u \\
&= \int_{\frac{1}{2}}^{2} \frac{u-1}{u^2\left(1+\frac{1}{u^2}\right)^{\frac{1}{2}}} \, \mathrm{d}u \\
&= \int_{\frac{1}{2}}^{2} \frac{u-1}{u\left(1+u^2\right)^{\frac{1}{2}}} \, \mathrm{d}u \\
&= -I.
\end{aligned}$$

This therefore means

$$I = \int_{\frac{1}{2}}^{2} \frac{1-x}{x\left(1+x^2\right)^{\frac{1}{2}}} \, \mathrm{d}x = 0.$$