2021.3 Question 8

1. We show this by induction on n.

   We first consider the base case where $n = 1$. Notice LHS $= x_1 = a$, and

   $$\text{RHS} = 2 + 4^{1-1}(a-2) = 2 + (a-2) = a.$$

   Hence, LHS $\geq$ RHS is true.

   Now, assume that the original statement

   $$x_n \geq 2 + 4^{n-1}(a-2)$$

   is true for some $n = k$.

   Consider the case where $n = k + 1$. We first notice that since $a > 2$, we must have

   $$x_n \geq 2 + 4^{n-1}(a-2) > 0.$$

   Hence, we have

   $$\begin{aligned}\text{LHS} &= x_{k+1} \\ &= x_k^2 - 2 \\ &\geq \left(2 + 4^{k-1}(a-2)\right)^2 - 2 \\ &= 4 + 4^{2k-2}(a-2)^2 + 4 \cdot 4^{k-1}(a-2) - 2 \\ &= 2 + 4^k(a-2) + 4^{2k-2}(a-2)^2 \\ &> 2 + 4^{(k+1)-1}(a-2) \\ &= \text{RHS},\end{aligned}$$

   and this shows that the original statement is true for the case $n = k + 1$ as well.

   Hence, the original statement is true for the base case $n = 1$, and given it holds for $n = k$, it holds for $n = k + 1$. By the principle of mathematical induction, it must hold for all integers $n \geq 1$ given $a > 2$, as desired.

2. - **If direction.** We are given that $|a| > 2$. If $a < 0$, we must have $a < -2$, but notice that for $x_1 = a$, $x_2 = a^2 - 2$, and for $x_1 = -a$, $x_2 = (-a)^2 - 2 = a^2 - 2$. Hence, if the first term only differs by a plus/minus sign, all the terms including and after the second term will behave identically. This means we only have to consider the case $a > 2$, and since

     $$x_n \geq 2 + 4^{n-1}(a-2),$$

     and the right-hand side diverges to ∞ as $n \to \infty$, we can conclude that

     $$\lim_{n\to\infty} x_n = \infty,$$

     as desired.

   - **Only-if direction.** We attempt to prove the contrapositive of the only-if direction, i.e. given that $|a| \leq 2$, we want to show that x_n does not diverge to ∞.

     We would like to show that $|x_n| \leq 2$ for all $n \in \mathbb{N}$.

     The base case where $n = 1$ is true, since $0 \leq a \leq 2$.

     Now, assume that this is true for some $n = k$, i.e.

     $$|x_n| \leq 2 \iff -2 \leq x_n \leq 2 \iff 0 \leq x_n^2 \leq 4.$$

     For $n = k + 1$,

     $$x_n = x_{k+1} = x_k^2 - 2,$$

     and hence

     $$-2 \leq x_{k+1} \leq 2 \iff |x_{k+1}| \leq 2.$$

So this statement is true for the base case where $n = 1$, and given it holds for some $n = k$ it holds for the case $n = k+1$. Hence, by the principle of mathematical induction, this statement is true for all $n \in \mathbb{N}$.

This means that x_n is bounded above and below, and hence it cannot diverge to infinity. This proves the contrapositive of the only-if direction, and hence the only-if direction is true.

In conclusion, we have shown that $x_n \to \infty$ as $n \to \infty$ if and only if $|a| > 2$.

3. If this is true for all $n \geq 1$, then this is true for $n = 1$. On one hand,

$$y_1 = \frac{Ax_1}{x_2} = \frac{Aa}{a^2 - 2},$$

and on the other hand

$$y_1 = \frac{\sqrt{x_2^2 - 4}}{x_2} = \frac{\sqrt{(a^2-2)^2 - 4}}{a^2 - 2} = \frac{\sqrt{a^4 - 4a^2}}{a^2 - 2} = \frac{a\sqrt{a^2 - 4}}{a^2 - 2}.$$

Hence, we must have

$$\begin{aligned} A &= \sqrt{a^2 - 4} \\ A^2 &= a^2 - 4 \\ a^2 &= A^2 + 4 \\ a &= \sqrt{A^2 + 4}, \end{aligned}$$

since $a > 2$.

We still have to show that this a gives the desired relation for every $n \geq 1$.

Notice that by definition,

$$\begin{aligned} y_{n+1} &= \frac{A \prod_{i=1}^{n+1} x_i}{x_{n+2}} \\ &= \frac{A \prod_{i=1}^{n}}{x_{n+1}} \cdot \frac{x_{n+1}^2}{x_{n+2}} \\ &= y_n \cdot \frac{x_{n+1}^2}{x_{n+2}}. \end{aligned}$$

We aim to show this by induction on n. The base case where $n = 1$ is shown above.

Now, assume that

$$y_n = \frac{\sqrt{x_{n+1}^2 - 4}}{x_{n+1}}$$

for a certain value of $n = k$.

For $n = k + 1$,

$$\begin{aligned} y_n &= y_{k+1} \\ &= y_k \cdot \frac{x_{n+1}^2}{x_{n+2}} \\ &= frac\sqrt{x_{n+1}^2 - 4}x_{n+1} \cdot \frac{x_{n+1}^2}{x_{n+2}} \\ &= \frac{\sqrt{x_{n+1}^2 - 4}x_{n+1}}{x_{n+2}} \\ &= \frac{\sqrt{x_{n+1}^4 - 4x_{n+1}^2}}{x_{n+2}} \\ &= \frac{\sqrt{\left(x_{n+1}^2 - 2\right)^2 - 4}}{x_{n+2}} \\ &= \frac{\sqrt{x_{n+2}^2 - 4}}{x_{n+2}}, \end{aligned}$$

which is precisely the original statement for $n = k + 1$.

By the principle of mathematical induction, for $a = \sqrt{A^2 + 4}$, we have shown that this desired statement holds for the base case $n = 1$, and given that it holds for some $n = k$, we can show it holds for $n = k + 1$. Hence, by the principle of mathematical induction, we have that

$$y_n = \frac{\sqrt{x_{n+1}^2 - 4}}{x_{n+1}}$$

for every value of $n \geq 1$ for this certain value of $a = \sqrt{A^2 + 4}$.

Hence, for the value $a = \sqrt{A^2 + 4}$, we have the statement holds for all $n \geq 1$. We have also shown that if the statement holds for all $n \geq 1$, it must be the case that $a = \sqrt{A^2 + 4}$. Hence, for precisely this value of $a = \sqrt{A^2 + 4}$, we have

$$y_n = \frac{\sqrt{x_{n+1}^2 + 4}}{x_{n+1}}.$$

For this value of $a > 2$, we have $x_n \to \infty$ as $n \to \infty$. Hence,

$$y_n = \frac{\sqrt{x_{n+1}^2 + 4}}{x_{n+1}} = \sqrt{1 + \frac{4}{x_{n+1}^2}}$$

converges to 1 as $n \to \infty$.