2021.3 Question 7

1. Notice that

$$\begin{aligned}
z &= \frac{\exp(i\theta)+\exp(i\varphi)}{\exp(i\theta)-\exp(i\varphi)} \\
&= \frac{\exp(i\theta)+\exp(i\varphi)}{\exp(i\theta)-\exp(i\varphi)} \cdot \frac{\exp(-i\theta)-\exp(-i\varphi)}{\exp(-i\theta)-\exp(-i\varphi)} \\
&= \frac{1+\exp(i\varphi-i\theta)-\exp(i\theta-i\varphi)-1}{1-\exp(i\theta-i\varphi)-\exp(i\varphi-i\theta)+1} \\
&= \frac{\exp(i(\varphi-\theta))-\exp(-i(\varphi-\theta))}{2-\exp(-i(\varphi-\theta))-\exp(i(\varphi-\theta))} \\
&= \frac{2i\sin(\varphi-\theta)}{2-2\cos(\varphi-\theta)} \\
&= \frac{i\sin(\varphi-\theta)}{1-\cos(\varphi-\theta)} \\
&= \frac{i\cdot 2\sin\frac{\varphi-\theta}{2}\cos\frac{\varphi-\theta}{2}}{1-(1-2\sin^2\frac{\varphi-\theta}{2})} \\
&= \frac{2i\sin\frac{\varphi-\theta}{2}\cos\frac{\varphi-\theta}{2}}{2\sin^2\frac{\varphi-\theta}{2}} \\
&= i\cot\frac{\varphi-\theta}{2},
\end{aligned}$$

as desired.

The modulus of z is $\left|\cot\frac{\varphi-\theta}{2}\right|$. The argument of z is $\pm\frac{\pi}{2}$.

2. Let $a=\exp(i\alpha)$, and $b=\exp(i\beta)$, where $a-b\neq 2n\pi$ for integer n (this ensures that A and B are distinct). We must have $x=a+b=\exp(i\alpha)+\exp(i\beta)$, and $b-a=\exp(i\beta)-\exp(i\alpha)$.

The vectors representing the two complex numbers are perpendicular, if and only if their argument differ by $\pm\frac{\pi}{2}$, if and only if their ratio has argument $\pm\frac{\pi}{2}$. Notice that the ratios

$$\begin{aligned}
\frac{OX}{AB} &= \frac{a+b}{b-a} \\
&= \frac{\exp(i\alpha)+\exp(i\beta)}{\exp(i\beta)-\exp(i\alpha)}
\end{aligned}$$

takes the same form as z before, and hence has argument $\pm\frac{\pi}{2}$. This hence means OX is perpendicular to AB.

3. Similarly, let $a=\exp(i\alpha)$, $b=\exp(i\beta)$, and $c=\exp(i\gamma)$, where no pair of α,β and γ differ by some multiple of 2π (which ensures that A,B,C are distinct points).

If H is the orthocentre of triangle ABC, then

$$h=a+b+c=\exp(i\alpha)+\exp(i\beta)+\exp(i\gamma),$$

and hence

$$\begin{aligned}
AH &= h-a=b+c=\exp(i\beta)+\exp(i\gamma), \\
BC &= c-b=\exp(i\gamma)-\exp(i\beta).
\end{aligned}$$

If $h\neq a$, then $AH=b+c\neq 0$, then the angle between AH and BC is given by the argument of the ratio of the complex numbers representing them, and notice

$$\frac{AH}{BC}=\frac{\exp(i\beta)+\exp(i\gamma)}{\exp(i\gamma)-\exp(i\beta)},$$

which takes the same form of z in the first part. Hence, the argument of this must be $\pm\frac{\pi}{2}$ since $b+c\neq 0$, which shows that AH is perpendicular to BC.

This means that either $h=a$, or AH is perpendicular to BC, as desired.

4. Similarly, let $a = \exp(i\alpha)$, $b = \exp(i\beta)$, $c = \exp(i\gamma)$ and $d = \exp(i\delta)$, where no pair of α, β, γ and δ differ by some multiple of 2π (which ensures that A, B, C, D are distinct points). Hence,

$$q = b + c + d = \exp(i\beta) + \exp(i\gamma) + \exp(i\delta),$$

and the midpoint of AQ, M, represented by complex number m, is given by

$$m = \frac{a + b + c + d}{2}.$$

By symmetry, the midpoint of BR, CS and DP must also be M.

This means that by an enlargement of scale factor -1 about M, A will be transformed to Q, B to R, C to S, and D to P.

Hence, $ABCD$ is transformed to $PQRS$ by an enlargement of scale factor -1, with centre of enlargement being $\frac{a+b+c+d}{4}$, the midpoint of AQ.