2021.3 Question 6

1. By multiplying by $\cot\alpha$ on top and bottom of the fraction, we have

$$\begin{aligned} f_\alpha(x) &= \arctan\left(\frac{x+\cot\alpha}{1-x\cot\alpha}\right) \\ &= \arctan\left(\frac{x+\tan\left(\frac{\pi}{2}-\alpha\right)}{1-x\tan\left(\frac{\pi}{2}-\alpha\right)}\right) \\ &= \arctan\tan\left(\arctan x+\frac{\pi}{2}-\alpha\right). \end{aligned}$$

Since $\arctan x \in \left(-\frac{\pi}{2},\alpha\right)\cup\left(\alpha,\frac{\pi}{2}\right)$, we have

$$\arctan x+\frac{\pi}{2}-\alpha \in \left(-\alpha,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi-\alpha\right).$$

Hence, we can simplify this to

$$\begin{aligned} f_\alpha(x) &= \arctan\tan\left(\arctan x+\frac{\pi}{2}-\alpha\right) \\ &= \begin{cases} \arctan x+\frac{\pi}{2}-\alpha, & x<\tan\alpha, \\ \arctan x-\frac{\pi}{2}-\alpha, & x>\tan\alpha. \end{cases} \end{aligned}$$

Hence, by differentiating with respect to x, the constants differentiate to 0, and hence

$$\begin{aligned} f'_\alpha(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\arctan x \\ &= \frac{1}{1+x^2}, \end{aligned}$$

as desired.

The graph consists of 2 branches of arctan, as the simplified expressions suggests. We have the following limiting behaviours of f_α:

$$\begin{aligned} \lim_{x\to-\infty} f_\alpha(x) &= \lim_{x\to-\infty}\arctan x+\frac{\pi}{2}-\alpha=-\alpha, \\ \lim_{x\to\tan\alpha^-} f_\alpha(x) &= \frac{\pi}{2}, \\ \lim_{x\to\tan\alpha^+} f_\alpha(x) &= -\frac{\pi}{2}, \\ \lim_{x\to\infty} f_\alpha(x) &= \lim_{x\to\infty}\arctan x-\frac{\pi}{2}-\alpha=-\alpha, \end{aligned}$$

which shows that f_α has a horizontal asymptote with equation $y=-\alpha$.

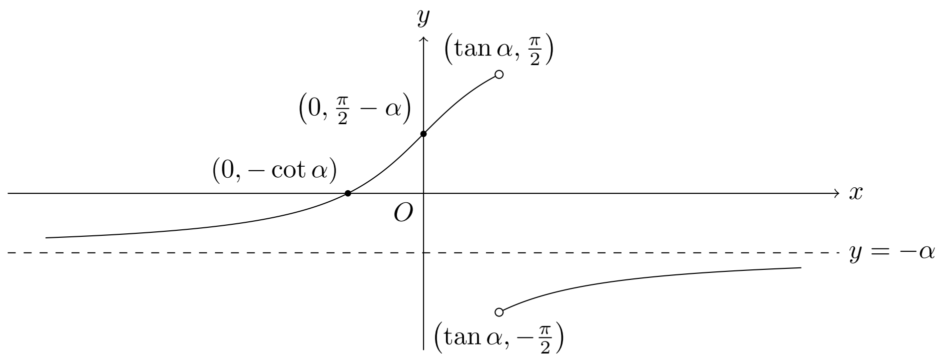
For the intersection with the y-axis,

$$f_\alpha(0)=\arctan 0+\frac{\pi}{2}-\alpha=\frac{\pi}{2}-\alpha,$$

and for the intersection with the x-axis,

$$f_\alpha(x)=0 \iff x\tan\alpha+1=0 \iff x=-\cot\alpha.$$

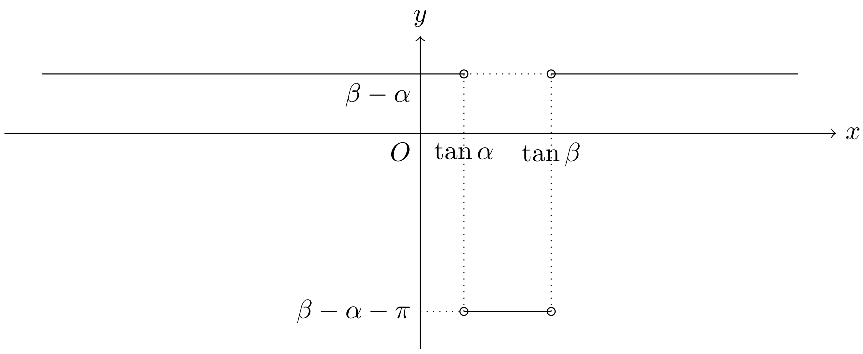
The graph looks as follows.



The domain of this new graph is $x \in \mathbb{R} \backslash \{\tan \alpha, \tan \beta\}$. By considering the functions in the different corresponding ranges, we have

$$f_\alpha(x) - f_\beta(x) = \begin{cases} \left(\arctan(x) + \frac{\pi}{2} - \alpha\right) - \left(\arctan(x) + \frac{\pi}{2} - \beta\right) = \beta - \alpha, & x < \tan \alpha, \\ \left(\arctan(x) - \frac{\pi}{2} - \alpha\right) - \left(\arctan(x) + \frac{\pi}{2} - \beta\right) = \beta - \alpha - \pi, & \tan \alpha < x < \tan \beta, \\ \left(\arctan(x) - \frac{\pi}{2} - \alpha\right) - \left(\arctan(x) - \frac{\pi}{2} - \beta\right) = \beta - \alpha, & \tan \beta < x. \end{cases}$$

Hence, the graph looks as follows.



2. By differentiation, we have

$$\begin{aligned} g'(x) &= \frac{1}{1 - \sin^2 x} \cos x - \frac{1}{\sqrt{1 + \tan^2 x}} \sec^2 x \\ &= \frac{\cos x}{\cos^2 x} - \frac{\sec^2 x}{|\sec x|} \\ &= \sec x - |\sec x| \\ &= \begin{cases} \sec x - \sec x = 0, & 0 \leq x < \frac{1}{2}\pi \text{ or } \frac{3}{2}\pi < x \leq 2\pi, \\ \sec x - (-\sec x) = 2 \sec x, & \frac{1}{2}\pi < x < \frac{3}{2}\pi, \end{cases} \end{aligned}$$

since $\sec x$ takes the same sign as $\cos x$, which is negative when $\frac{1}{2}\pi < x < \frac{3}{2}\pi$, and positive when $0 \leq x < \frac{1}{2}\pi$ or $\frac{3}{2}\pi < x \leq 2\pi$ within the range.
For $\frac{1}{2}\pi < x < \frac{3}{2}\pi$, we must have

$$g(x) = \ln|\tan x + \sec x| + C = \ln(-\tan x - \sec x) + C,$$

and by verifying

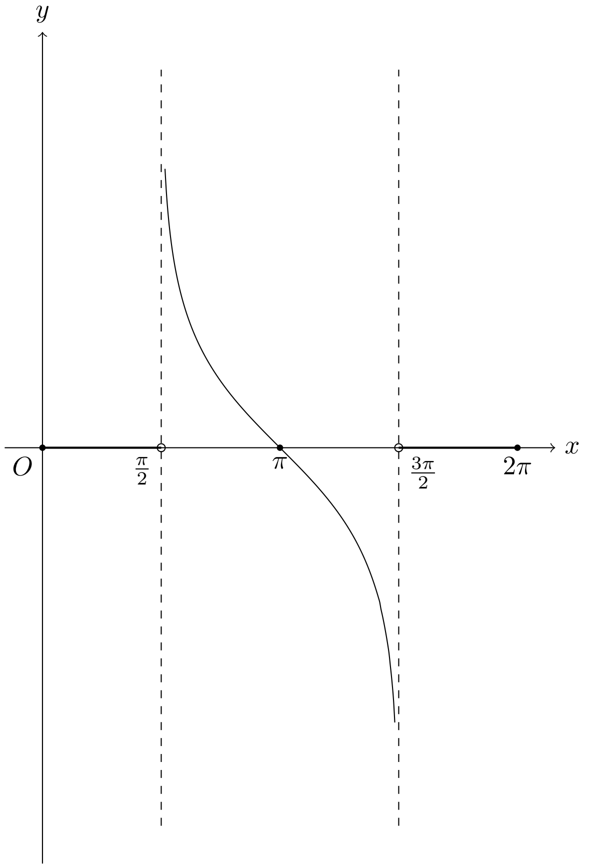
$$g(\pi) = \operatorname{artanh}(0) - \operatorname{arsinh}(0) = 0,$$

we can see $C = 0$.
Hence, for $0 \leq x < \frac{1}{2}\pi$ and $\frac{3}{2}\pi < x \leq 2\pi$ respectively, $g(x)$ is constant, and notice that

$$g(0) = g(2\pi) = 0,$$

and hence

$$g(x) = \begin{cases} \ln(-\tan x - \sec x), & \frac{1}{2}\pi < x < \frac{3}{2}\pi, \\ 0, & 0 \leq x < \frac{1}{2}\pi \text{ or } \frac{3}{2}\pi \leq 2\pi. \end{cases}$$


y
x
O
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π