2021.3 Question 5

1. When the curves meet, the r values and the θ values must be both equal, and hence

$$\begin{aligned}a + 2\cos\theta &= 2 + \cos 2\theta \\ a + 2\cos\theta &= 2 + 2\cos^2\theta - 1 \\ 2\cos^2\theta - 2\cos\theta + 1 - a &= 0,\end{aligned}$$

as desired.

By differentiating with respect to theta, for the two curves to touch, we must have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}(a + 2\cos\theta) &= \frac{\mathrm{d}}{\mathrm{d}\theta}(2 + \cos 2\theta) \\ -2\sin\theta &= -2\sin 2\theta \\ \sin\theta &= \sin 2\theta \\ \sin\theta &= 2\sin\theta\cos\theta \\ \sin\theta(2\cos\theta - 1) &= 0.\end{aligned}$$

This means, either for the value of $\sin\theta = 0$ it satisfies the first equation, or for the value of $2\cos\theta - 1 = 0$ it satisfies the first equation.

For the first case, we must have $\cos\theta = \pm 1$, and hence

$$\begin{aligned}a &= 2\cos^2\theta - 2\cos\theta + 1 \\ &= 2(\pm 1)^2 - 2(\pm 1) + 1 \\ &= 3 \pm 2,\end{aligned}$$

and so $a = 1$ or $a = 5$.

For the second case, we have $\cos\theta = \frac{1}{2}$, and hence

$$\begin{aligned}a &= 2\cos^2\theta - 2\cos\theta + 1 \\ &= 2\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) + 1 \\ &= \frac{1}{2},\end{aligned}$$

as desired.

2. For the case where $a = \frac{1}{2}$, the curves meet precisely for $\cos\theta = \frac{1}{2}$ only, and hence $\theta = \pm\frac{\pi}{3}$, which gives $r = \frac{1}{2} + 1 = \frac{3}{2}$.

Both curves are symmetric about the initial line, since cos is an even function.

When $\theta = 0$, $r_1 = a + 2 = \frac{5}{2}$, and $r_2 = 2 + 1 = 3$.

For r_1, since $r \geq 0$, we must have

$$\begin{aligned}\frac{1}{2} + 2\cos\theta &\geq 0 \\ \cos\theta &\geq -\frac{1}{4},\end{aligned}$$

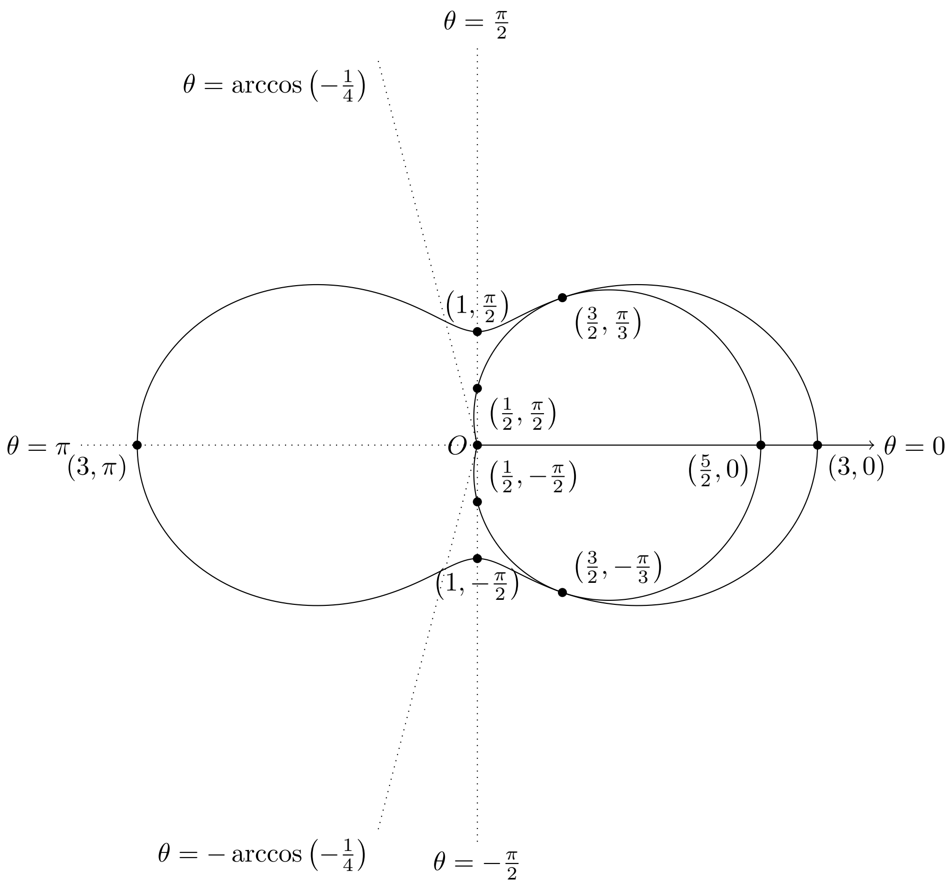
which means it only exists for

$$-\arccos\left(-\frac{1}{4}\right) \leq \theta \leq \arccos\left(-\frac{1}{4}\right).$$

When $\theta = \pm\frac{\pi}{2}$, $r_1 = \frac{1}{2} + 2\cos\pm\frac{\pi}{2} = \frac{1}{2}$.

For all values of θ, we must have $r_2 \geq 0$. When $\theta = \pi$, $r_2 = 2 + 1 = 3$, and for $\theta = \pm\frac{\pi}{2}$, $r_1 = \frac{1}{2} + \cos\pm\frac{\pi}{2} = \frac{1}{2}$, $r_2 = 2 + \cos\pm\pi = 1$.

Hence, the two curves are as follows. All coordinates are in (r, θ).



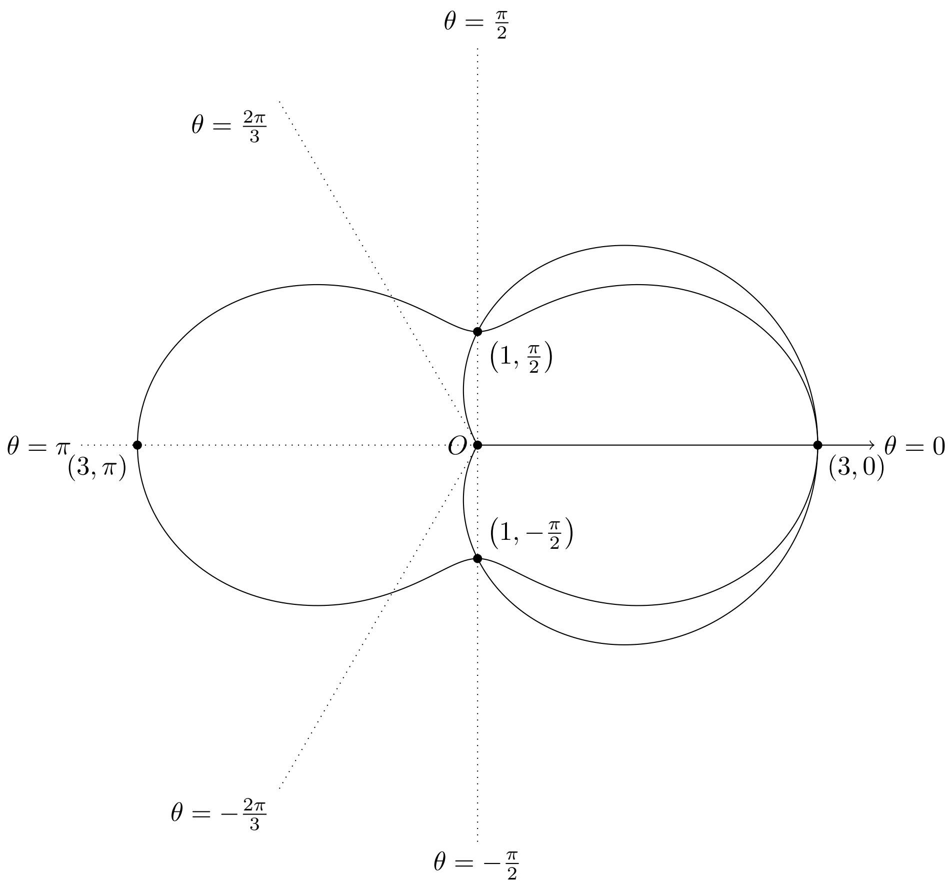
3. • $a = 1$. For r_1, since $r \geq 0$, we must have

$$\begin{aligned} 1 + 2\cos\theta &\geq 0 \\ \cos\theta &\geq -\frac{1}{2}, \end{aligned}$$

which means $-\frac{2}{3}\pi \leq \theta \leq \frac{2}{3}\pi$.
The two curves meet when

$$\begin{aligned} 2\cos^2\theta - 2\cos\theta &= 0 \\ \cos\theta(\cos\theta - 1) &= 0, \end{aligned}$$

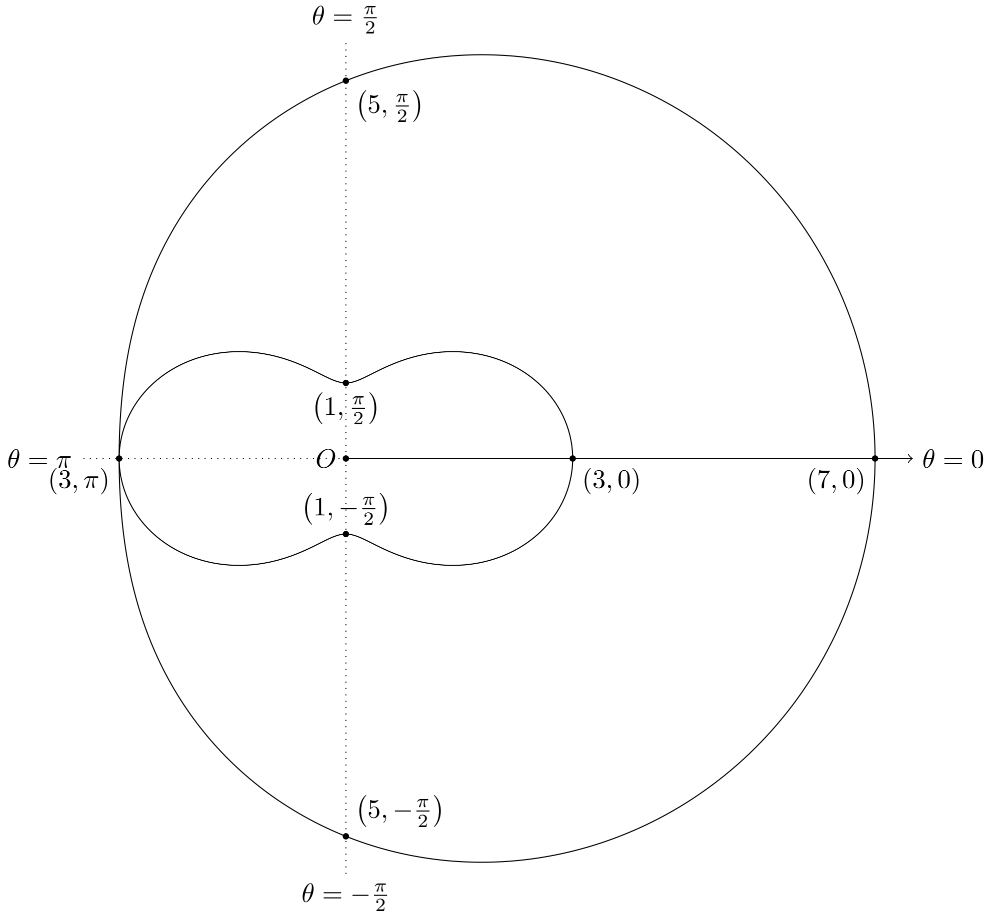
which is when $\cos\theta = 0$ or $\cos\theta = 1$.
For $\cos\theta = 0$, this means $\theta = \pm\frac{\pi}{2}$, and $r = 1$. For this value of θ, the two curves cross.
For $\cos\theta = 1$, this means $\theta = 0$, and $r = 3$. For this value of θ, the two curves touch.



- $a = 5$. For r_1, $r \geq 0$ for all θ.
  The two curves meet when

$$\begin{aligned} 2\cos^2\theta - 2\cos\theta &= 4 \\ \cos^2\theta - \cos\theta - 2 &= 0 \\ (\cos\theta - 2)(\cos\theta + 1) &= 0, \end{aligned}$$

  which is when $\cos\theta = -1$, since $\cos\theta \neq 2$.
  For $\cos\theta = -1$, this means $\theta = \pi$, and $r = 3$. For this value of θ, the two curves touch.
  When $\theta = 0$, $r_1 = 5 + 2 = 7$, and $r_2 = 2 + 1 = 3$. When $\theta = \pm\frac{1}{2}\pi$, $r_1 = 5 + 2\cos\pm\frac{1}{2}\pi = 5$, $r_2 = 2 + \cos\pm\pi = 1$.



$\theta = \frac{\pi}{2}$
$\left(5, \frac{\pi}{2}\right)$
$\left(1, \frac{\pi}{2}\right)$
$\theta = \pi$
$(3, \pi)$
O
$(3, 0)$
$(7, 0)$
$\theta = 0$
$\left(1, -\frac{\pi}{2}\right)$
$\left(5, -\frac{\pi}{2}\right)$
$\theta = -\frac{\pi}{2}$