2021.3 Question 4

1. Since θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, we have

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \mathbf{a}\cdot\mathbf{b}.$$

Let λ be the angle between $\mathbf{m}$ and $\mathbf{a}$. Hence,

$$\begin{aligned}\cos\lambda &= \frac{\mathbf{a}\cdot\mathbf{m}}{|\mathbf{a}||\mathbf{m}|} \\ &= \frac{\mathbf{a}\cdot\frac{1}{2}(\mathbf{a}+\mathbf{b})}{|\mathbf{m}|} \\ &= \frac{\mathbf{a}\cdot(\mathbf{a}+\mathbf{b})}{|\mathbf{a}+\mathbf{b}|} \\ &= \frac{1+\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}+\mathbf{b}|} \\ &= \frac{1+\cos\theta}{|\mathbf{a}+\mathbf{b}|}.\end{aligned}$$

Similarly, let μ be the angle between $\mathbf{m}$ and $\mathbf{b}$, and we must have

$$\cos\lambda = \cos\mu = \frac{1+\cos\theta}{|\mathbf{a}+\mathbf{b}|}.$$

Since $0 \leq \lambda, \mu \leq \pi$, and cos is one-to-one when restricted to $[0, \pi]$, we must have $\lambda = \mu$, which shows that $\mathbf{m}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

2. We must have $\cos\alpha = \mathbf{a}\cdot\mathbf{c}$, and $\cos\beta = \mathbf{b}\cdot\mathbf{c}$.

By definition of the projection, we have

$$\begin{aligned}\mathbf{a}_1 &= \mathbf{a} - (\mathbf{a}\cdot\mathbf{c})\,\mathbf{c} \\ &= \mathbf{a} - \cos\alpha\mathbf{c},\end{aligned}$$

and hence

$$\begin{aligned}\mathbf{a}_1\cdot\mathbf{c} &= \mathbf{a}\cdot\mathbf{c} - \cos\alpha\mathbf{c}\cdot\mathbf{c} \\ &= \cos\alpha - \cos\alpha \\ &= 0,\end{aligned}$$

as desired.

Notice that

$$\begin{aligned}|\mathbf{a}_1|^2 &= \mathbf{a}_1\cdot\mathbf{a}_1 \\ &= (\mathbf{a} - \cos\alpha\mathbf{c})\cdot(\mathbf{a} - \cos\alpha\mathbf{c}) \\ &= \mathbf{a}\cdot\mathbf{a} - 2\cos\alpha\mathbf{a}\cdot\mathbf{c} + \cos^2\alpha\mathbf{c}\cdot\mathbf{c} \\ &= 1 - 2\cos^2\alpha + \cos^2\alpha \\ &= 1 - \cos^2\alpha \\ &= \sin^2\alpha.\end{aligned}$$

Since $|a_1| \geq 0$, and $0 < \alpha < \frac{\pi}{2}$, $\sin\alpha > 0$, we must have

$$|\mathbf{a}_1| = |\sin\alpha| = \sin\alpha.$$

The angle φ is given by

$$\begin{aligned}\cos\varphi &= \frac{\mathbf{a}_1\cdot\mathbf{b}_1}{|\mathbf{a}_1||\mathbf{b}_1|}\\ &= \frac{(\mathbf{a}-\cos\alpha\mathbf{c})\cdot(\mathbf{b}-\cos\beta\mathbf{c})}{\sin\alpha\sin\beta}\\ &= \frac{\mathbf{a}\cdot\mathbf{b}-\cos\alpha\mathbf{b}\cdot\mathbf{c}-\cos\beta\mathbf{a}\cdot\mathbf{c}+\cos\alpha\cos\beta\mathbf{c}\mathbf{c}}{\sin\alpha\sin\beta}\\ &= \frac{\cos\theta-\cos\alpha\cos\beta-\cos\beta\cos\alpha+\cos\beta\cos\alpha}{\sin\alpha\sin\beta}\\ &= \frac{\cos\theta-\cos\alpha\cos\beta}{\sin\alpha\sin\beta}.\end{aligned}$$

3. By definition of a projection, we have

$$\begin{aligned}\mathbf{m}_1 &= \mathbf{m}-(\mathbf{m}\cdot\mathbf{c})\mathbf{c}\\ &= \frac{1}{2}(\mathbf{a}+\mathbf{b})-\left(\frac{1}{2}(\mathbf{a}+\mathbf{b})\cdot\mathbf{c}\right)\mathbf{c}\\ &= \frac{1}{2}(\mathbf{a}+\mathbf{b})-\left(\frac{1}{2}(\cos\alpha+\cos\beta)\right)\mathbf{c}\\ &= \frac{1}{2}(\mathbf{a}_1+\mathbf{b}_1).\end{aligned}$$

Let ν be the angle between $\mathbf{m}_1$ and $\mathbf{a}_1$, we have

$$\begin{aligned}\cos\nu &= \frac{\mathbf{m}_1\cdot\mathbf{a}_1}{|\mathbf{m}_1||\mathbf{a}_1|}\\ &= \frac{\frac{1}{2}(\mathbf{a}_1+\mathbf{b}_1)\cdot\mathbf{a}_1}{\frac{1}{2}|\mathbf{a}_1+\mathbf{b}_1|\sin\alpha}\\ &= \frac{\mathbf{a}_1\cdot\mathbf{a}_1+\mathbf{b}_1\cdot\mathbf{a}_1}{|\mathbf{a}_1+\mathbf{b}_1|\sin\alpha}\\ &= \frac{\sin^2\alpha+\cos\varphi\sin\alpha\sin\beta}{|\mathbf{a}_1+\mathbf{b}_1|\sin\alpha}\\ &= \frac{\sin^2\alpha+\cos\theta-\cos\alpha\cos\beta}{|\mathbf{a}_1+\mathbf{b}_1|\sin\alpha}.\end{aligned}$$

Similarly, let τ be the angle between $\mathbf{m}_1$ and $\mathbf{b}_1$, we have

$$\cos\tau = \frac{\sin^2\beta+\cos\theta-\cos\alpha\cos\beta}{|\mathbf{a}_1+\mathbf{b}_1|\sin\beta}.$$

Since $0\leq\nu,\tau\leq\pi$, $\nu=\tau$ if and only if

$$\begin{aligned}\cos\nu &= \cos\tau\\ \frac{\sin^2\alpha+\cos\theta-\cos\alpha\cos\beta}{|\mathbf{a}_1+\mathbf{b}_1|\sin\alpha} &= \frac{\sin^2\beta+\cos\theta-\cos\alpha\cos\beta}{|\mathbf{a}_1+\mathbf{b}_1|\sin\beta}\\ \sin\beta\left(\sin^2\alpha+\cos\theta-\cos\alpha\cos\beta\right) &= \sin\alpha\left(\sin^2\beta+\cos\theta-\cos\alpha\cos\beta\right)\\ \sin\alpha\sin\beta(\sin\alpha-\sin\beta)+\cos\alpha\cos\beta(\sin\alpha-\sin\beta) &= \cos\theta(\sin\alpha-\sin\beta)\\ (\sin\alpha\sin\beta+\cos\alpha\cos\beta)(\sin\alpha-\sin\beta) &= \cos\theta(\sin\alpha-\sin\beta)\\ (\cos(\alpha-\beta)-\cos\theta)(\sin\alpha-\sin\beta) &= 0.\end{aligned}$$

This is if and only if $\sin\alpha=\sin\beta$, or $\cos\theta=\cos(\alpha-\beta)$.

Since $0<\alpha,\beta<\frac{\pi}{2}$, and sin is one-to-one when restricted to $\left(0,\frac{\pi}{2}\right)$, the first condition is true if and only if $\alpha=\beta$.

Hence, $\mathbf{m}_1$ bisects the angle between $\mathbf{a}_1$ and $\mathbf{b}_1$ if and only if $\alpha=\beta$ or $\cos\theta=\cos(\alpha-\beta)$, as desired.