2021.3 Question 3

1. Notice that

$$\begin{aligned}
\text{LHS} &= \frac{1}{2}\left(I_{n+1} + I_{n-1}\right) \\
&= \frac{1}{2}\left(\int_0^\beta (\sec x + \tan x)^{n+1}\, \mathrm{d}x + \int_0^\beta (\sec x + \tan x)^{n-1}\, \mathrm{d}x\right) \\
&= \frac{1}{2}\int_0^\beta (\sec x + \tan x)^{n-1}\left[(\sec x + \tan x)^2 + 1\right] \mathrm{d}x \\
&= \frac{1}{2}\int_0^\beta (\sec x + \tan x)^{n-1}\left(\sec^2 x + \tan^2 x + 2\sec x \tan x + 1\right) \mathrm{d}x \\
&= \frac{1}{2}\int_0^\beta (\sec x + \tan x)^{n-1} \cdot 2\left(\sec^2 x + \sec x \tan x\right) \mathrm{d}x \\
&= \int_0^\beta (\sec x + \tan x)^{n-1}\, \mathrm{d}(\sec x + \tan x) \\
&= \frac{1}{n}\left[(\sec x + \tan x)^n\right]_0^\beta \\
&= \frac{1}{n}\left((\sec \beta + \tan \beta)^n - (\sec 0 + \tan 0)^n\right) \\
&= \frac{1}{n}\left((\sec \beta + \tan \beta)^n - 1\right) \\
&= \text{RHS},
\end{aligned}$$

as desired.

To show the final part, we would like to show that

$$I_n < \frac{1}{2}\left(I_{n+1} + I_{n-1}\right) = \frac{1}{n}\left((\sec \beta + \tan \beta)^n - 1\right),$$

which is equivalent to showing

$$I_{n+1} + I_{n-1} - 2I_n > 0.$$

$$\begin{aligned}
&I_{n+1} + I_{n-1} - 2I_n \\
&= \int_0^\beta (\sec x + \tan x)^{n+1}\, \mathrm{d}x + \int_0^\beta (\sec x + \tan x)^{n-1}\, \mathrm{d}x - 2\int_0^\beta (\sec x + \tan x)^n\, \mathrm{d}x \\
&= \int_0^\beta (\sec x + \tan x)^{n-1}\left(2\sec^2 x + 2\sec x \tan x - 2\sec x - 2\tan x\right) \\
&= \int_0^\beta (\sec x + \tan x)^{n-1}\left(\sec^2 x + \tan^2 x + 2\sec x \tan x - 2\sec x - 2\tan x + 1\right) \\
&= \int_0^\beta (\sec x + \tan x)^{n-1}\left[(\sec x + \tan x)^2 - 2\left(\sec x + 2\tan x\right) + 1\right] \\
&= \int_0^\beta (\sec x + \tan x)^{n-1}\left((\sec x + \tan x) - 1\right)^2.
\end{aligned}$$

For $0 \leq x < \frac{\pi}{2}$, $\sec x > 0, \tan x > 0$, and so $\sec x + \tan x > 0$, $(\sec x + \tan x)^{n-1} > 0$.

Also, $\sec x = \frac{1}{\cos x} > \frac{1}{1} = 1$, and hence $\sec x + \tan x - 1 > 0$, so $((\sec x + \tan x) - 1)^2 > 0$.

Hence, the integrand is greater than 0 on $(0, \beta) \subseteq \left(0, \frac{\pi}{2}\right)$.

This shows that the desired equation is greater than 0, and hence, we have the desired inequality as desired.

2. Notice that

$$\begin{aligned}\frac{1}{2}\left(J_{n+1}+J_{n-1}\right) &= \frac{1}{2}\int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left[(\sec x\cos\beta+\tan x)^2+1\right]\mathrm{d}x\\ &= \frac{1}{2}\int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left[\sec^2 x\cos^2\beta+\tan^2 x+2\sec x\tan x\cos\beta+1\right]\mathrm{d}x\\ &= \frac{1}{2}\int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left(\sec^2 x\cos^2\beta+\sec^2 x+2\sec x\tan x\cos\beta\right)\mathrm{d}x\\ &= \frac{1}{2}\int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left(2\sec^2 x-\sec^2 x\sin^2\beta+2\sec x\tan x\cos\beta\right)\mathrm{d}x\\ &= \int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left(\sec^2 x+\sec x\tan x\cos\beta\right)\mathrm{d}x\\ &\quad -\frac{\sin^2\beta}{2}\int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\sec^2 x\,\mathrm{d}x.\end{aligned}$$

The first part of the integral integrates similarly:

$$\begin{aligned}&\int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left(\sec^2 x+\sec x\tan x\cos\beta\right)\mathrm{d}x\\ &= \int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\,\mathrm{d}\left(\sec x\cos\beta+\tan x\right)\\ &= \frac{1}{n}\left[(\sec x\cos\beta+\tan x)^n\right]_0^\beta\\ &= \frac{1}{n}\left[(\sec\beta\cos\beta+\tan\beta)^n-(\sec 0\cos\beta+\tan 0)^n\right]\\ &= \frac{1}{n}\left[(1+\tan\beta)^n-\cos^n\beta\right].\end{aligned}$$

The second part of the integral has a positive integrand over $(0,\beta)$, and hence the integral is positive, which means

$$\begin{aligned}\frac{1}{2}\left(J_{n+1}+J_{n-1}\right) &> \int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left(\sec^2 x+\sec x\tan x\cos\beta\right)\mathrm{d}x\\ &= \frac{1}{n}\left[(1+\tan\beta)^n-\cos^n\beta\right].\end{aligned}$$

We would like to show that $J_{n+1}+J_{n-1}-2J_n>0$ similar as before to show the final result. Note that

$$\begin{aligned}&J_{n+1}+J_{n-1}-2J_n\\ &= \int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left[(\sec x\cos\beta+\tan x)^2+1-2\left(\sec x\cos\beta+\tan x\right)\right]\mathrm{d}x\\ &= \int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left[(\sec x\cos\beta+\tan x)-1\right]^2\mathrm{d}x\\ &> 0,\end{aligned}$$

and hence $J_n<\frac{1}{2}\left(J_{n+1}+J_{n-1}\right)$, which shows

$$J_n<\frac{1}{n}\left((1+\tan\beta)^n-\cos^n\beta\right),$$

as desired.