2021.3 Question 2

1. For the first row/component in $\hat{\mathbf{\imath}}$,

$$\begin{aligned}\begin{pmatrix}1 & -x & x\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} &= 1\cdot a + (-x)\cdot b + x\cdot c\\ &= a + \frac{-ab}{b-c} = \frac{ac}{b-c}\\ &= a + \frac{ac-ab}{b-c}\\ &= a + (-a)\\ &= 0,\end{aligned}$$

and this is similar for the remaining row and components. Hence, we have

$$\begin{pmatrix}1 & -x & x\\ y & 1 & -y\\ -z & z & 1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

as desired.

If the matrix

$$\begin{pmatrix}1 & -x & x\\ y & 1 & -y\\ -z & z & 1\end{pmatrix}$$

is invertible, then we must have

$$\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}1 & -x & x\\ y & 1 & -y\\ -z & z & 1\end{pmatrix}^{-1}\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

which is impossible, since a, b and c are distinct.

Hence, this matrix is not invertible, and it must have a zero-determinant, meaning

$$\begin{aligned}0 &= \det\begin{pmatrix}1 & -x & x\\ y & 1 & -y\\ -z & z & 1\end{pmatrix}\\ &= 1\cdot 1\cdot 1 + (-x)\cdot(-y)\cdot(-z) + x\cdot y\cdot z - 1\cdot(-y)\cdot z - (-x)\cdot y\cdot 1 - x\cdot 1\cdot(-z)\\ &= 1 - xyz + xyz + yz + xy + xz\\ &= xy + yz + zx + 1,\end{aligned}$$

and hence

$$xy + yz + zx = -1.$$

Since $(x+y+z)^2 \geq 0$, we have

$$\begin{aligned}0 &\leq (x+y+z)^2\\ &= x^2 + y^2 + z^2 + 2(xy+yz+zx)\\ &= \frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} + 2\cdot(-1),\end{aligned}$$

and hence

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} \geq 2,$$

as desired.

2. Consider the matrix
$$\begin{pmatrix} -2 & x & x \\ y & -2 & y \\ z & z & -2, \end{pmatrix}$$
and for the first row/component in $\hat{\mathbf{\imath}}$,
$$\begin{aligned} \begin{pmatrix} -2 & x & x \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= (-2)a + bx + cx \\ &= (-2)a + (b+c)x \\ &= (-2)a + 2a \\ &= 0, \end{aligned}$$
and similarly in the second and third rows/components, and hence
$$\begin{pmatrix} -2 & x & x \\ y & -2 & y \\ z & z & -2, \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$
By similar argument as before, this matrix must have a zero determinant as well, and hence
$$\begin{aligned} 0 &= \det \begin{pmatrix} -2 & x & x \\ y & -2 & y \\ z & z & -2, \end{pmatrix} \\ &= (-2)(-2)(-2) + xyz + xyz - (-2)yz - xy(-2) - x(-2)z \\ &= -8 + 2xyz + 2xy + 2yz + 2zx, \end{aligned}$$
and hence
$$xyz + xy + yz + zx = 4,$$
as desired.

Hence, consider
$$(x+1)(y+1)(z+1) = xyz + xy + yz + zx + x + y + z + 1 = 5 + x + y + z.$$
Since a, b, c are all positive real numbers, x, y, z are as well, and hence $x + y + z > 0$, giving
$$(x+1)(y+1)(z+1) > 5,$$
which means
$$\frac{2a+b+c}{b+c} \cdot \frac{a+2b+c}{a+c} \cdot \frac{a+b+2c}{a+b} > 5,$$
and hence
$$(2a+b+c)(a+2b+c)(a+b+2c) > 5(b+c)(c+a)(a+b)$$
as desired.

Furthermore, notice that
$$\begin{aligned} x + y + z &= \frac{2a}{b+c} + \frac{2b}{c+a} + \frac{2c}{a+b} \\ &> \frac{2a}{a+b+c} + \frac{2b}{a+b+c} + \frac{2c}{a+b+c} \\ &= \frac{2(a+b+c)}{a+b+c} \\ &= 2. \end{aligned}$$
Hence,
$$(x+1)(y+1)(z+1) > 7,$$

which means

$$\frac{2a+b+c}{b+c} \cdot \frac{a+2b+c}{a+c} \cdot \frac{a+b+2c}{a+b} > 7,$$

and hence

$$(2a+b+c)(a+2b+c)(a+b+2c) > 7(b+c)(c+a)(a+b)$$

as desired.